

EN221 - Fall2008 - HW # 10 Solutions

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1.) (i) Show that, for a compressible Newtonian viscous fluid, the extra stress power $\text{tr}(\sigma^E \mathbf{D})$ is universally non-negative if and only if the bulk and shear viscosities are non negative.

(ii) An incompressible Newtonian viscous fluid occupies a regular region \mathbf{B} with a fixed rigid boundary $\partial\mathbf{B}$ and is not acted on by body forces. Show that the total kinetic energy of the fluid decreases at the rate.

$$-\mu \int_{\mathbf{B}} |\omega|^2 dv \quad (1)$$

μ being the viscosity and ω the vorticity.

Soln.

For a compressible Newtonian Viscous fluid

$$\begin{aligned} \sigma &= \left(-p + \left(\kappa - \frac{2}{3}\mu \right) \text{tr} \mathbf{D} \right) \mathbf{I} + 2\mu \mathbf{D} \\ &= -p\mathbf{I} + \sigma^E \\ \sigma^E &= \left(\kappa - \frac{2}{3}\mu \right) \text{tr}(\mathbf{D}) \mathbf{I} + 2\mu \mathbf{D} \end{aligned}$$

The extra stress power is

$$\begin{aligned} \text{tr}(\sigma^E \mathbf{D}) &= \text{tr} \left[\left(\kappa - \frac{2}{3}\mu \right) \text{tr}(\mathbf{D}) \mathbf{D} \right] + \text{tr}[2\mu \mathbf{D}^2] \\ &= \left(\kappa - \frac{2}{3}\mu \right) (\text{tr} \mathbf{D})^2 + 2\mu \text{tr}(\mathbf{D}^2) \end{aligned}$$

writing, $\mathbf{D} = \mathbf{D}' + \frac{1}{3}(\text{tr} \mathbf{D})\mathbf{I}$, ($\text{tr} \mathbf{D}' = 0$, by definition)
we obtain

$$\begin{aligned} \text{tr}(\sigma^E \mathbf{D}) &= \left(\kappa - \frac{2}{3}\mu \right) (\text{tr} \mathbf{D})^2 + 2\mu \text{tr} \left([\mathbf{D}' + \frac{1}{3}(\text{tr} \mathbf{D})\mathbf{I}]^2 \right) \\ [\mathbf{D}' + \frac{1}{3}(\text{tr} \mathbf{D})\mathbf{I}]^2 &= \mathbf{D}'^2 + \frac{2}{3}(\text{tr} \mathbf{D})\mathbf{D}' + \frac{1}{9}(\text{tr} \mathbf{D})^2 \mathbf{I} \\ \text{tr} [\mathbf{D}' + \frac{1}{3}(\text{tr} \mathbf{D})\mathbf{I}]^2 &= \text{tr}(\mathbf{D}'^2) + \frac{1}{3}(\text{tr} \mathbf{D})^2 \\ \Rightarrow \text{tr}(\sigma^E \mathbf{D}) &= \left(\kappa - \frac{2}{3}\mu \right) (\text{tr} \mathbf{D})^2 + 2\mu \text{tr}(\mathbf{D}'^2) + \frac{2\mu}{3}(\text{tr} \mathbf{D})^2 \end{aligned}$$

$$\Rightarrow \text{tr} (\sigma^{\mathbf{E}} \mathbf{D}) = \kappa (\text{tr} \mathbf{D})^2 + 2\mu \text{tr} (\mathbf{D}'^2) \quad (2)$$

since $(\text{tr} \mathbf{D})^2 \geq 0$ and $\text{tr} (\mathbf{D}'^2) \geq 0$. hence, $\text{tr} (\sigma^{\mathbf{E}} \mathbf{D}) \geq 0$ if κ, μ are non-negative

(ii) The total KE is

$$KE = \frac{1}{2} \int_B \rho v^2 dV \Rightarrow \frac{d KE}{dt} = \int_B \rho v \cdot \dot{v} dV \quad (3)$$

But

$$\rho \dot{v} = \text{div} \sigma + \rho b + \mu \nabla^2 v \quad (4)$$

$\nabla \cdot v = 0$, incompressible fluid

$$\rho \dot{v} = -\nabla p + \mu \nabla^2 v \quad (5)$$

Thus,

$$\frac{d KE}{dt} = - \int_B \nabla p \cdot v dV + \mu \int_B \nabla^2 v \cdot v dV \quad (6)$$

now, $\nabla \cdot (pv) = \nabla p \cdot v + p \nabla \cdot v = \nabla p \cdot v$ so; Using the divergence theorem

$$\int_B \nabla p \cdot v dV = \int_B \nabla \cdot (pv) dV = \int_{\partial B} pv \cdot n dA = 0 \quad (7)$$

But since the boundary is fixed $v \cdot n = 0$

$$\nabla^2 v = \nabla (\nabla \cdot v) - \nabla \wedge (\nabla \wedge v) = -\nabla \wedge (\nabla \wedge v) \quad (8)$$

$$\int_B \mu \nabla^2 v \cdot v dV = - \int_B \mu \nabla \wedge (\nabla \wedge v) \cdot v dV \quad (9)$$

but, $\nabla \wedge v = \omega$

$$\int_B \mu \nabla^2 v \cdot v dV = - \int_B \mu (\nabla \wedge \omega) \cdot v dV \quad (10)$$

But,

$$\begin{aligned} (\nabla \wedge \omega) \cdot v &= \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j} v_i \\ &= \epsilon_{ijk} \frac{\partial (\omega_k v_i)}{\partial x_j} - \epsilon_{ijk} \omega_k \frac{\partial v_i}{\partial x_j} \\ &= \epsilon_{ijk} \frac{\partial -(\omega \wedge v)_j}{\partial x_j} - \epsilon_{ijk} \omega_k \frac{\partial v_i}{\partial x_j} \\ &= -\nabla \cdot (\omega \wedge v) + (\nabla \wedge v) \cdot \omega \\ &= -\nabla \cdot (\omega \wedge v) + \omega \cdot \omega \end{aligned} \quad (11)$$

Thus,

$$\int_B \mu \nabla^2 v \cdot v \, dV = \mu \int_B \nabla \cdot (\omega \wedge v) \, dV - \mu \int_B |w|^2 \, dV \quad (12)$$

$$\int_B \nabla \cdot (\omega \wedge v) \, dV = \int_{\partial B} (\omega \wedge v) \cdot n \, dA = 0 \quad \text{since } v \cdot n = 0 \quad (13)$$

$$\int_B \mu \nabla^2 v \cdot v \, dV = -\mu \int_B |w|^2 \, dV \quad (14)$$

2.) (a) Show that the circulatory motion defined in Exercise 2.9 pg 85 Chadwick, can be maintained in an incompressible Newtonian viscous fluid without the aid of body forces if the function f satisfies the partial differential equation

$$\nu \left(\frac{\partial^2 f}{\partial r^2} + \frac{3}{r} \frac{\partial f}{\partial r} \right) = \frac{\partial f}{\partial t} \quad (15)$$

and that the pressure is then given by

$$p = p_0 + \rho \int^r \{f(s, t)\}^2 s ds \quad (16)$$

Here ρ is the density and ν the kinematic viscosity of the fluid and p_0 is a constant.

(b) An incompressible Newtonian fluid is confined between long coaxial rigid cylinders of radii r_1 and r_2 ($r_1 < r_2$) which rotate about their common axis with angular speeds Ω_1 and Ω_2 respectively. Assuming that the fluid performs a steady circulatory motion and that body forces are absent, deduce from the results of part (a) that the circumferential velocity at distance r from the axis is

$$\frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2} r - \frac{\Omega_2 - \Omega_1}{r_2^2 - r_1^2} \frac{r_1^2 r_2^2}{r} \quad (17)$$

Calculate the torque per unit length needed to drive each cylinder.

Soln.

$$v_1 = -x_2 f(r, t) = -r \sin \theta f(r, t) \quad (18)$$

$$v_2 = x_1 f(r, t) = r \cos \theta f(r, t) \quad (19)$$

$$v_3 = 0 \quad (20)$$

$$\Rightarrow \mathbf{v} = f(r, t) r e_\theta \quad (21)$$

$$\Rightarrow \rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{b} \quad (22)$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \text{grad } \mathbf{v} \cdot \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} \quad (23)$$

$$\begin{aligned} \text{grad } \mathbf{v} &= \left(e_r \frac{\partial}{\partial r} + \frac{e_\theta}{r} \frac{\partial}{\partial \theta} + e_z \frac{\partial}{\partial z} \right) \otimes f(r, t) r e_\theta \\ &= \frac{\partial r f}{\partial r} e_\theta \otimes e_r + r f \frac{\partial e_\theta}{\partial \theta} \otimes e_\theta \\ &= \frac{\partial r f}{\partial r} e_\theta \otimes e_r - r f e_r \otimes e_\theta \end{aligned} \quad (24)$$

$$\begin{aligned} \text{grad } \mathbf{v} \cdot \mathbf{v} &= \left(\frac{\partial r f}{\partial r} e_\theta \otimes e_r - r f e_r \otimes e_\theta \right) \cdot (f r e_\theta) \\ &= -f r^2 e_r \end{aligned} \quad (25)$$

$$-\nabla p = - \left(e_r \frac{\partial p}{\partial r} + \frac{e_\theta}{r} \frac{\partial p}{\partial \theta} + e_z \frac{\partial p}{\partial z} \right) \quad (26)$$

Note that, \mathbf{v} is function of only r and t so $\frac{\partial}{\partial \theta} = 0 = \frac{\partial}{\partial z}$ by the symmetry in the problem

$$\begin{aligned}
\nabla^2 \mathbf{v} &= \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right] f r e_\theta \\
&= \left[\frac{e_\theta}{r} \frac{\partial}{\partial r} r \frac{\partial f r}{\partial r} + \frac{f}{r} \frac{\partial^2 e_\theta}{\partial \theta^2} \right] \\
&= \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial f r}{\partial r} - \frac{f}{r} \right] e_\theta
\end{aligned} \tag{27}$$

On substituting Eqn(25 to 27) in Eqn(23)

$$\begin{aligned}
\rho \left[\frac{\partial r f}{\partial t} e_\theta - f^2 r e_r \right] &= -e_r \frac{\partial p}{\partial r} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial f r}{\partial r} - \frac{f}{r} \right) e_\theta \\
\Rightarrow \rho \frac{\partial r f}{\partial t} &= \mu \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial f r}{\partial r} - \frac{f}{r} \right)
\end{aligned} \tag{28}$$

$$\Rightarrow f^2 r = \frac{\partial p}{\partial r} \tag{29}$$

Using Eqn(28) on simplification

$$\Rightarrow \frac{\partial f}{\partial t} = \frac{\mu}{\rho} \left(\frac{\partial^2 f}{\partial r^2} + \frac{3}{r} \frac{\partial f}{\partial r} \right) \tag{30}$$

Using Eqn(29) on integrating

$$\begin{aligned}
\frac{\partial p}{\partial r} &= f^2 r \\
p &= \int^r f^2 s ds + p_O
\end{aligned} \tag{31}$$

(b) Fluid motion is steady

$$\begin{aligned}
\frac{\partial f}{\partial t} &= 0 \\
\Rightarrow \frac{d^2 f}{dr^2} + \frac{3}{r} \frac{df}{dr} &= 0
\end{aligned} \tag{32}$$

The general solution of Eqn(32) is give by

$$\Rightarrow f = -\frac{\beta}{2r^2} + \gamma \tag{33}$$

now $\mathbf{v}] f r e_\theta$

$$V(R_1) = \Omega_1 R_1 = \left(-\frac{\beta}{2R_1^2} + \gamma \right) R_1$$

$$V(R_2) = \Omega_2 R_2 = \left(-\frac{\beta}{2R_2^2} + \gamma \right) R_2$$

$$\Rightarrow \Omega_1 = -\frac{\beta}{2R_1^2} + \gamma \quad (34)$$

$$\Rightarrow \Omega_2 = -\frac{\beta}{2R_2^2} + \gamma \quad (35)$$

$$\Rightarrow \Omega_1 - \Omega_2 = \frac{\beta}{2} \left(\frac{1}{R_2^2} - \frac{1}{R_1^2} \right)$$

$$\Rightarrow \beta = 2 \frac{\Omega_1 - \Omega_2}{R_1^2 - R_2^2} (R_1 R_2)^2 \quad (36)$$

$$\Rightarrow \omega_1 = -\frac{\Omega_1 - \Omega_2}{R_1^2 - R_2^2} \frac{(R_1 R_2)^2}{R_1^2} + \gamma$$

$$\Rightarrow \gamma = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \quad (37)$$

$$\Rightarrow v = fr = \left(\frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \right) r - \frac{\Omega_1 - \Omega_2}{R_1^2 - R_2^2} \frac{(R_1 R_2)^2}{r^2} \quad (38)$$

The torque will be created by the shear stress $\sigma_{r\theta}$ on the cylinder.

$$\sigma = -p\mathbf{I} + 2\mu\mathbf{D} \quad (39)$$

$$\mathbf{L} = \text{grad}\mathbf{v} = \frac{\partial r f}{\partial r} e_\theta \otimes e_r - r f e_r \otimes e_\theta \quad (40)$$

using $v_\theta = fr$

$$\mathbf{L} = \frac{\partial v_\theta}{\partial r} e_\theta \otimes e_r - \frac{v_\theta}{r} e_r \otimes e_\theta \quad (41)$$

$$\mathbf{D} = \frac{1}{2} \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) (e_\theta \otimes e_r + e_r \otimes e_\theta) \quad (42)$$

$$\Rightarrow \sigma_{r\theta} = \mu \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) e_\theta \otimes e_r \quad (43)$$

$$\text{Torque } T = \sigma_{r\theta} * 2\pi * r * r = 2\mu\pi r^2 \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) \quad (44)$$

using the v_θ obtained earlier

$$T = \frac{2\mu R_1^2 R_2^2}{r^2} \left(\frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \right) * 2 * \pi r^2$$

$$T = \frac{4\pi\mu R_1^2 R_2^2}{r^2} \left(\frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \right) \quad (45)$$

T is constant, which is not surprising because if we draw a annular cylindrical control volume; the angular momentum within this volume does not change with time. so the torque is constant.

3). Show that the constitutive equation of an incompressible Reiner-Rivlin fluid can be expressed in the form

$$\sigma = -p\mathbf{I} + \nu_1\mathbf{D} + \nu_2\mathbf{D}^2 \quad (46)$$

where the response function ν_1 and ν_2 depend upon the principal stretching invariants II_D and III_D .

Such a fluid fills the space between parallel rigid plates. One plate is held fixed and an steady shearing motion is produced in the fluid by translating the other plate in its own plane with constant speed \mathcal{V} . Introducing a system of rectangular Cartesian coordinates x_1, x_2, x_3 in which the stationary and moving plates occupy the planes $x_2 = 0$ and $x_2 = d$ respectively and the moving plate travels in the 1-direction and assuming that the velocity field in the fluid is of the form

$$v_1 = v(x_2), \quad v_2 = 0 \quad v_3 = 0 \quad (47)$$

calculate the stress components. Verify that the shear stress σ_{12} is a function (τ , say) of D_{12} only. Given that there is no pressure gradient in the 1-direction, that no body forces act, and that τ is single-valued, show that $v_1 = Vx_2/d$ and p is constant.

Specialize your results to the case in which the fluid is Newtonian. In what main respect do the two solutions differ?

Soln.

$$\sigma = (-p + \nu_o)\mathbf{I} + \nu_1\mathbf{D} + \nu_2\mathbf{D}^2 \quad (48)$$

since the fluid is incompressible, the constraint is,

$$\lambda(\mathbf{C}) = \det\mathbf{C} - 1 \quad (\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2) \quad (49)$$

as shown on pg 146-147 of Chadwick

$$\dot{\lambda}(\mathbf{C}) = (\det\mathbf{C})\text{tr } \dot{\mathbf{C}}\mathbf{C}^{-1} = \text{tr } \mathbf{C}^{-1}\dot{\mathbf{C}} \quad (50)$$

also, $\dot{\lambda} = \text{tr } \lambda_{\mathbf{C}}\dot{\mathbf{C}}$

$$\Rightarrow \lambda_{\mathbf{C}} = \frac{\partial\lambda}{\partial\mathbf{C}} = \mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-1^T} \quad (51)$$

The constraint stress is hence, $N = \alpha\mathbf{F}\lambda_{\mathbf{C}}\mathbf{F}^T = \alpha\mathbf{I}$ where α is some constant. The effective stress is

$$\begin{aligned} \sigma &= (-p + \nu_o)\mathbf{I} + \nu_1\mathbf{D} + \nu_2\mathbf{D}^2 + \alpha\mathbf{I} \\ &= (-p + \nu_o + \alpha)\mathbf{I} + \nu_1\mathbf{D} + \nu_2\mathbf{D}^2 \end{aligned} \quad (52)$$

α is independent of \mathbf{D} or ρ (i.e. not determined by the configuration of the body),

It has to be determined only via equilibrium equations .It is hence legitimate to call $-p + \nu_o + \alpha = -P$ (effective stress)

$$\begin{aligned} v_1 &= V(x_2) \\ v_2 &= 0 \\ v_3 &= 0 \end{aligned}$$

$$\mathbf{L} = \text{grad } \mathbf{v} = \frac{\partial V}{\partial x_2} e_1 \otimes e_2 \quad (53)$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \frac{1}{2} \frac{\partial V}{\partial x_2} (e_1 \otimes e_2 + e_2 \otimes e_1) \quad (54)$$

$$\mathbf{D}^2 = \frac{1}{2} \left(\frac{\partial V}{\partial x_2} \right)^2 (e_1 \otimes e_1 + e_2 \otimes e_2) \quad (55)$$

$$\sigma = -P\mathbf{I} + \nu_1 \mathbf{D} + \nu_2 \mathbf{D}^2$$

$$\Rightarrow \sigma = -P\mathbf{I} + \nu_1 \left[\frac{1}{2} \frac{\partial V}{\partial x_2} (e_1 \otimes e_2 + e_2 \otimes e_1) \right] + \nu_2 \left[\frac{1}{2} \left(\frac{\partial V}{\partial x_2} \right)^2 (e_1 \otimes e_1 + e_2 \otimes e_2) \right]$$

$$\sigma_{12} = \frac{\nu_1}{2} \frac{\partial V}{\partial x_2} e_1 \otimes e_2 = \nu_1 D_{12} \quad (56)$$

$I_D = 0$, $II_D = -D_{12}^2$, and $III_D = 0$, hence ν_1 and ν_2 are just functions of D_{12}

since $\sigma_{12} = \frac{\nu_1}{2} \frac{\partial V}{\partial x_2} e_1 \otimes e_2 = \nu_1 (D_{12}^2) D_{12}$

σ_{12} is only function of D_{12}

Now,

$$\begin{aligned} \rho \dot{\mathbf{v}} &= \nabla \cdot \sigma + \rho \mathbf{b} \\ \rho \left(\frac{\partial V}{\partial t} + \text{grad } \mathbf{v} \cdot \mathbf{v} \right) &= \nabla \cdot \sigma \\ \Rightarrow 0 &= \nabla \cdot \sigma \quad \text{since } \frac{\partial V}{\partial t} = 0 \quad \text{and} \quad \text{grad } \mathbf{v} \cdot \mathbf{v} = 0 \end{aligned}$$

Since $\frac{\partial P}{\partial x_1}$ and

$\frac{d\nu_1 D_{12}}{dx_2} = 0 \Rightarrow \nu_2(D_{12}) D_{12} = \tau(\text{Constant in space})$

Note that $D_{12} = \beta(\text{constant in space})$

would mean

$$\nu_1(D_{12}^2) = \nu_1(\beta^2) = \gamma(\text{a constant}) \quad (57)$$

$$\Rightarrow \tau = \beta\gamma \quad (58)$$

if $D_{12} = \beta$

$$\begin{aligned} \Rightarrow \frac{dV}{dx_2} &= \beta \\ \Rightarrow V &= \beta x_2 + \delta \end{aligned} \quad (59)$$

from no slip conditions $V(0) = 0$ and $V(d) = \mathcal{V} \Rightarrow V = \frac{\mathcal{V}}{d}x_2$ is a solution
 $\Rightarrow D_{12} = \frac{\mathcal{V}}{d}$

$\text{div}\sigma = 0$ in 2-direction

$$\Rightarrow \frac{d}{dx_2} \left(-P + \frac{\nu_2(D_{12}^2)}{4} \left(\frac{dV}{dx_2} \right)^2 \right) = 0 \quad (60)$$

$$\Rightarrow P = -P_0 + \frac{\nu_2(D_{12}^2)}{4} \left(\frac{dV}{dx_2} \right)^2 \quad (61)$$

$$\Rightarrow P = -P_0 + \frac{\nu_2}{4} \left(\frac{\mathcal{V}}{d} \right)^2 \quad (62)$$

Thus the pressure is constant

hence, $V = \frac{\mathcal{V}}{d}x_2$ and $P = \text{Constant}$ is one solution

there might be another solution if $\nu_1(D_{12}^2)$ has some particular form $\nu_2(D_{12}^2)D_{12} = \tau \Rightarrow \text{if } \nu_2 = \frac{\tau}{D_{12}}$

Then we obtain $\nu_2(D_{12}^2)D_{12} = \text{constant}$

This means that any velocity field and D_{12} will satisfy the $\text{div}\sigma = 0$, equation, which is not a physical possibility. Similarly, the conditions of velocity and pressure are independent of the nature of the fluid and should hold universally for any $\nu_2(D_{12}^2)$. This implies that $D_{12} = \frac{\mathcal{V}}{d}$ or $V = \frac{\mathcal{V}}{d}x_2$ is the only solution.

if the fluid is non-Newtonian $\nu_2 \left(\frac{\mathcal{V}}{d} \right) = 0$ and $\nu_1(D_{12}^2) = \mu$

It can be easily seen that $\sigma_{11} = \sigma_{12} = P_o$

so, in Non-linear fluids, we need additional stress $\frac{1}{4} \left(\frac{\mathcal{V}}{d} \right)^2 \nu_2 \left(\left(\frac{\mathcal{V}}{d} \right)^2 \right)$ to keep the motion. Note that this is a second order effect in $\left(\frac{\mathcal{V}}{d} \right)$

4). (i) if I_1 , I_2 and I_3 are principal invariants of the left stretch tensor \mathbf{V} , prove the following identities,

$$\frac{\partial I_1}{\partial \mathbf{V}} = \mathbf{I}, \quad \frac{\partial I_2}{\partial \mathbf{V}} = I \operatorname{tr} \mathbf{V} - \mathbf{V}, \quad \frac{\partial I_3}{\partial \mathbf{V}} = \det(\mathbf{V}) \mathbf{V}^{-1} \quad (63)$$

(ii) In class we showed that the Cauchy stress can be expressed as

$$\mathbf{T} = \phi_0 \mathbf{I} + \phi_1 \mathbf{V} + \phi_2 \mathbf{V}^2 \quad (64)$$

where ϕ_i are functions of the principal invariants of \mathbf{V} . Show that the Cauchy stress can also be expressed as

$$\mathbf{T} = \psi_0 \mathbf{I} + \psi_1 \mathbf{V}^2 + \psi_2 \mathbf{V}^4 \quad (65)$$

and derive expressions for ψ_i 's in terms of ϕ_i 's and the principal invariants of \mathbf{V} .

(Hint use Cayley-Hamilton theorem)

Soln.

$$\mathbf{V} = \sum_{i=1}^3 e_i \otimes e_i \quad \lambda_i \text{ are eigenvalues of } \mathbf{V} \quad (66)$$

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3 \quad (67)$$

$$\Rightarrow \frac{\partial I_1}{\partial V} = \sum_{i=1}^3 \frac{\partial I_1}{\partial \lambda_i} e_i \otimes e_i = \sum_{i=1}^3 e_i \otimes e_i = \mathbf{I} \quad (68)$$

$$I_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 \quad (69)$$

$$\begin{aligned} \Rightarrow \frac{\partial I_2}{\partial V} &= \sum_{i=1}^3 \frac{\partial I_2}{\partial \lambda_i} e_i \otimes e_i \\ &= (\lambda_1 + \lambda_2) e_3 \otimes e_3 + (\lambda_1 + \lambda_3) e_2 \otimes e_2 + (\lambda_2 + \lambda_3) e_1 \otimes e_1 \\ &= I(\lambda_1 + \lambda_2 + \lambda_3) - \sum \lambda_i e_i \otimes e_i \\ &= I \operatorname{tr} \mathbf{V} - \mathbf{V} \end{aligned} \quad (70)$$

$$I_3 = \lambda_1 \lambda_2 \lambda_3 \quad (71)$$

$$\begin{aligned} \Rightarrow \frac{\partial I_3}{\partial V} &= \lambda_2 \lambda_3 e_1 \otimes e_1 + \lambda_3 \lambda_1 e_2 \otimes e_2 + \lambda_1 \lambda_2 e_3 \otimes e_3 \\ &= \lambda_1 \lambda_2 \lambda_3 \left(\frac{1}{\lambda_1} e_1 \otimes e_1 + \frac{1}{\lambda_2} e_2 \otimes e_2 + \frac{1}{\lambda_3} e_3 \otimes e_3 \right) \\ &= \det \mathbf{V} \mathbf{V}^{-1} \end{aligned} \quad (72)$$

(ii)

$$\mathbf{T} = \phi_0 \mathbf{I} + \phi_1 \mathbf{V} + \phi_2 \mathbf{V}^2 \quad (73)$$

using Cayley Hamilton theorem

$$\begin{aligned}\mathbf{V}^3 - \mathbf{I}_1 \mathbf{V}^2 + \mathbf{I}_2 \mathbf{V} - \mathbf{I}_3 \mathbf{I} &= 0 \\ \mathbf{V}^3 &= \mathbf{I}_1 \mathbf{V}^2 - \mathbf{I}_2 \mathbf{V} + \mathbf{I}_3 \mathbf{I}\end{aligned}\tag{74}$$

$$(75)$$

$$\begin{aligned}\mathbf{V}^4 &= \mathbf{I}_1 \mathbf{V}^3 - \mathbf{I}_2 \mathbf{V}^2 + \mathbf{I}_3 \mathbf{V} \\ &= \mathbf{I}_1 (\mathbf{I}_1 \mathbf{V}^2 - \mathbf{I}_2 \mathbf{V} + \mathbf{I}_3 \mathbf{I}) - \mathbf{I}_2 \mathbf{V}^2 + \mathbf{I}_3 \mathbf{V} \\ &= (\mathbf{I}_1^2 - \mathbf{I}_2) \mathbf{V}^2 + (-\mathbf{I}_1 \mathbf{I}_2 + \mathbf{I}_3) \mathbf{V} + \mathbf{I}_1 \mathbf{I}_3 \mathbf{I}\end{aligned}\tag{76}$$

$$\begin{aligned}\Rightarrow &= \psi_0 \mathbf{I} + \psi_1 \mathbf{V}^2 + \psi_2 \mathbf{V}^4 \\ &= \psi_0 \mathbf{I} + \psi_1 \mathbf{V}^2 + \psi_2 [(\mathbf{I}_1^2 - \mathbf{I}_2) \mathbf{V}^2 + (-\mathbf{I}_1 \mathbf{I}_2 + \mathbf{I}_3) \mathbf{V} + \mathbf{I}_1 \mathbf{I}_3 \mathbf{I}] \\ &= \phi_0 \mathbf{I} + \phi_1 \mathbf{V} + \phi_2 \mathbf{V}^2\end{aligned}\tag{77}$$

$$\phi_0 = \psi_0 + \psi_2 I_1 I_3\tag{78}$$

$$\phi_1 = \psi_1 + (I_1^2 - I_2) \psi_2\tag{79}$$

$$\phi_2 = \psi_2 (I_3 - I_1 I_2)\tag{80}$$

on Solving,

$$\psi_2 = \frac{\phi_1}{I_3 - I_1 I_2}\tag{81}$$

$$\psi_1 = \phi_2 - \frac{\phi_1 (I_1^2 - I_2)}{I_3 - I_1 I_2}\tag{82}$$

$$\psi_0 = \phi_0 - \frac{\phi_1 I_1 I_3}{I_3 - I_1 I_2}\tag{83}$$