EN221 - Fall2008 - HW # 1 Solutions

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1. Let $\mathbf{A}$ be an arbitrary tensor.

i). Show that

$$II_A = \frac{1}{2}\{(\text{tr } \mathbf{A})^2 - \text{tr } \mathbf{A^2}\}$$

ii). Using the Cayley-Hamilton theorem, deduce that

$$III_A = \frac{1}{6}\{(\text{tr } \mathbf{A})^3 - 3\,\text{tr } \mathbf{A} \text{ tr } \mathbf{A^2} + 2\,\text{tr } \mathbf{A^3}\}$$

Soln.

i). **Method-1**

Using the definition of $\mathbf{II_A}$, i.e. Eqn(1.26) from Chadwick

$$\mathbf{II_A}\,[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{a},\mathbf{Ab},\mathbf{Ac}] + [\mathbf{Aa},\mathbf{b},\mathbf{Ac}] + [\mathbf{Aa},\mathbf{Ab},\mathbf{c}] \tag{1}$$

Let, $\mathbf{a} = \mathbf{e_1}$, $\mathbf{b} = \mathbf{e_2}$ and $\mathbf{c} = \mathbf{e_3}$

$$\mathbf{II_A}[\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}] = [\mathbf{e_1},\mathbf{Ae_2},\mathbf{Ae_3}] + [\mathbf{Ae_1},\mathbf{e_2},\mathbf{Ae_3}] + [\mathbf{Ae_1},\mathbf{Ae_2},\mathbf{e_3}] \tag{2}$$

$$\mathbf{II_A}\epsilon_{\mathbf{123}} = [\mathbf{e_1},\mathbf{Ae_2},\mathbf{Ae_3}] + [\mathbf{Ae_1},\mathbf{e_2},\mathbf{Ae_3}] + [\mathbf{Ae_1},\mathbf{Ae_2},\mathbf{e_3}] \tag{3}$$

since, $\epsilon_{123} = 1$, and

$$\mathbf{Ae_i} = \mathbf{A_{mn}}\,(\mathbf{e_m}\otimes\mathbf{e_n})\mathbf{e_i} = \mathbf{A_{mn}}\,(\mathbf{e_n}\cdot\mathbf{e_i})\mathbf{e_m} = \mathbf{A_{mn}}\delta_{\mathbf{ni}}\mathbf{e_m} = \mathbf{A_{mi}}\mathbf{e_m}$$

$$\mathbf{II_A} = \mathbf{A_{q2}}\mathbf{A_{r3}}[\mathbf{e_1},\mathbf{e_q},\mathbf{e_r}] + \mathbf{A_{p1}}\mathbf{A_{r3}}[\mathbf{e_p},\mathbf{e_2},\mathbf{e_r}] + \mathbf{A_{p1}}\mathbf{A_{q2}}[\mathbf{e_p},\mathbf{e_q},\mathbf{e_3}]$$

since p, q and r are dummy indices (and rearranging)

$$\begin{aligned}\mathbf{II_A} &= \mathbf{A_{p2}}\mathbf{A_{q3}}[\mathbf{e_1},\mathbf{e_p},\mathbf{e_q}] + \mathbf{A_{p1}}\mathbf{A_{q3}}[\mathbf{e_p},\mathbf{e_2},\mathbf{e_q}] + \mathbf{A_{p1}}\mathbf{A_{q2}}[\mathbf{e_p},\mathbf{e_q},\mathbf{e_3}] \\ &= \mathbf{A_{p2}}\mathbf{A_{q3}}\epsilon_{\mathbf{1pq}} - \mathbf{A_{p1}}\mathbf{A_{q3}}\epsilon_{\mathbf{2pq}} + \mathbf{A_{p1}}\mathbf{A_{q2}}\epsilon_{\mathbf{3pq}} \end{aligned} \tag{4}$$

we can as well write it by interchanging p and q

$$\begin{aligned}\mathbf{II_A} &= \mathbf{A_{q2}}\mathbf{A_{p3}}\epsilon_{\mathbf{1qp}} - \mathbf{A_{q1}}\mathbf{A_{p3}}\epsilon_{\mathbf{2qp}} + \mathbf{A_{q1}}\mathbf{A_{p2}}\epsilon_{\mathbf{3qp}} \\ &= -\mathbf{A_{q2}}\mathbf{A_{p3}}\epsilon_{\mathbf{1pq}} + \mathbf{A_{q1}}\mathbf{A_{p3}}\epsilon_{\mathbf{2pq}} - \mathbf{A_{q1}}\mathbf{A_{p2}}\epsilon_{\mathbf{3pq}} \end{aligned} \tag{5}$$

on summing Eqn(4) and Eqn(5)

$$\begin{aligned} \mathbf{2\,II_A} &= (\mathbf{A_{p2}A_{q3}} - \mathbf{A_{q2}A_{p3}})\epsilon_{\mathbf{1pq}} + ... \\ &\quad -(\mathbf{A_{p1}A_{q3}} - \mathbf{A_{q1}A_{p3}})\epsilon_{\mathbf{2pq}} + ... \\ &\quad +(\mathbf{A_{p1}A_{q2}} - \mathbf{A_{q1}A_{p2}})\epsilon_{\mathbf{3pq}} \end{aligned} \tag{6}$$

$$\begin{aligned} \mathbf{2\,II_A} &= (\mathbf{A_{p2}A_{q3}}\epsilon_{\mathbf{123}} + \mathbf{A_{p3}A_{q2}}\epsilon_{\mathbf{132}})\epsilon_{\mathbf{1pq}} + ... \\ &\quad +(\mathbf{A_{p1}A_{q3}}\epsilon_{\mathbf{213}} + \mathbf{A_{p3}A_{q1}}\epsilon_{\mathbf{231}})\epsilon_{\mathbf{2pq}} + ... \\ &\quad +(\mathbf{A_{p1}A_{q2}}\epsilon_{\mathbf{312}} + \mathbf{A_{p2}A_{q1}}\epsilon_{\mathbf{321}})\epsilon_{\mathbf{3pq}} \end{aligned} \tag{7}$$

Eqn(7) can be re written as

$$\mathbf{II_A} = \frac{1}{2}\mathbf{A_{pi}A_{qj}}\epsilon_{\mathbf{rij}}\epsilon_{\mathbf{rpq}} \tag{8}$$

If we closely examine Eqn(8), we observe that r is different from {i,j} and the sign of ϵ_{rij} determines the sign of the each term. All the existing six terms of the Eqn(7) are obtained by the all the {i,j,r} permutations of the Eqn(8)

using the fact (see P1.3b Chadwick)

$$\epsilon_{rij}\epsilon_{rpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

Eqn(8) can be written as,

$$\begin{aligned} \mathbf{II_A} &= \frac{1}{2}\mathbf{A_{pi}A_{qj}}(\delta_{\mathbf{ip}}\delta_{\mathbf{jq}} - \delta_{\mathbf{iq}}\delta_{\mathbf{jp}}) \\ &= \frac{1}{2}(\mathbf{A_{ii}\,A_{jj}} - \mathbf{A_{ij}\,A_{ji}}) \\ &= \frac{1}{2}\{(\mathbf{tr\ A})^{\mathbf{2}} - \mathbf{tr\ A^2}\} \end{aligned} \tag{9}$$

Method-2

$$\mathbf{II_A}\,[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{a},\mathbf{Ab},\mathbf{Ac}] + [\mathbf{Aa},\mathbf{b},\mathbf{Ac}] + [\mathbf{Aa},\mathbf{Ab},\mathbf{c}] \tag{10}$$

we know that $\mathbf{A}^*\,(\mathbf{a}\wedge\mathbf{b}) = \mathbf{A\,a}\,\wedge\,\mathbf{A\,b}$ and also,
$\mathbf{B}\,(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c} = (\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{B}^{\mathbf{T}}\mathbf{c}$

$$\begin{aligned} \mathbf{II_A}\,[\mathbf{a},\mathbf{b},\mathbf{c}] &= [(\mathbf{A}^*)^{\mathbf{T}}\,\mathbf{a},\mathbf{b},\mathbf{c}] + [\mathbf{a},(\mathbf{A}^*)^{\mathbf{T}}\,\mathbf{b},\mathbf{c}] + [\mathbf{a},\mathbf{b},(\mathbf{A}^*)^{\mathbf{T}}\,\mathbf{c}] \\ &= \mathbf{tr}\,(\mathbf{A}^*)^{\mathbf{T}}[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{tr}\,\mathbf{A}^*\,[\mathbf{a},\mathbf{b},\mathbf{c}] \end{aligned} \tag{11}$$

Using Cayley-Hamilton theorem

$$\mathbf{A^3 - I_A\, A^2 + II_A\, A - III_A = 0} \tag{12}$$

Multiply above equation with A^{-1} to get,

$$\mathbf{A^2 - I_A\, A + II_A - A^{-1}\, III_A = 0} \tag{13}$$

knowing the fact that $III_A = detA$ and $A^* = A^{-1}detA$, the above Eqn. can be re-written as

$$\mathbf{A^2 - I_A\, A + II_A - A^* = 0} \tag{14}$$

Now, take the trace of the above Eqn.

$$\begin{aligned}
&\mathbf{tr\ A^2 - I_A\, tr\ A + II_A\, tr\ I - tr\ (A^*) = 0} \\
&\mathbf{tr\ A^2 - (tr\ A)^2 + 2\, II_A = 0} \\
&\mathbf{II_A = \frac{1}{2}\{(tr\ A)^2 - tr\ A^2\}}
\end{aligned} \tag{15}$$

ii) Cayley-Hamilton theorem states that,

$$\mathbf{A^3 - I_A\, A^2 + II_A\, A - III_A = 0} \tag{16}$$

Taking the trace of the above Eqn. we obtain,

$$\mathbf{tr\ A^3 - I_A\, tr\ A^2 + II_A\, tr\ A - III_A\, tr\ I = 0} \tag{17}$$

using the fact that $\mathbf{II_A = \frac{1}{2}\{(tr\ A)^2 - tr\ A^2\}}$ **and** $\mathbf{tr\ I = 3}$
we obtain,

$$\mathbf{tr\ A^3 - I_A\, tr\ A^2 + \frac{1}{2}\{(tr\ A)^2 - tr\ A^2\}\, tr\ A - 3\, III_A = 0} \tag{18}$$

on simplifying,

$$\begin{aligned} 3\,\mathbf{III_A} &= \mathbf{tr\ A^3 - tr\ A\, tr\ A^2 + \frac{1}{2}\{(tr\ A)^2 - tr\ A^2\}\, tr\ A} \\ &= \frac{1}{2}[\mathbf{2\, tr\ A^3 - 3\, tr\ A\, tr\ A^2 + (tr\ A)^3}] \end{aligned} \tag{19}$$

or

$$\mathbf{III_A} = \frac{1}{6}[\mathbf{2\, tr\ A^3 - 3\, tr\ A\ tr\ A^2 + (tr\ A)^3}] \tag{20}$$

2. Using the result

$$\mathbf{detA}\ \ \mathbf{detB} = \mathbf{det}(\mathbf{A}^{\mathbf{T}}\mathbf{B}) \qquad \forall\ \mathbf{A},\ \mathbf{B}\ \in\ \mathbf{L},$$

or otherwise, show that

$$\begin{aligned}\epsilon_{ijk}\epsilon_{lmn} &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) \\ &\quad +\delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\end{aligned}$$

Hence derive the formula,
(a) $\epsilon_{ijp}\epsilon_{lmp} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$,
(b) $\epsilon_{ipq}\epsilon_{lpq} = 2\,\delta_{il}$
Soln.

$$e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}) \tag{21}$$

eg. $e_1 = (\delta_{11}, \delta_{12}, \delta_{13}) = (1, 0, 0)$

$$\epsilon_{ijk} = [e_i, e_j, e_k] = \begin{vmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{vmatrix} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \tag{22}$$

$$\begin{aligned}\epsilon_{ijk}\ \epsilon_{lmn} &= \begin{vmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{vmatrix} \begin{vmatrix} \delta_{l1} & \delta_{m1} & \delta_{n1} \\ \delta_{l2} & \delta_{m2} & \delta_{n2} \\ \delta_{l3} & \delta_{m3} & \delta_{n3} \end{vmatrix} && (23) \\ &= \left| \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix} \begin{bmatrix} \delta_{l1} & \delta_{m1} & \delta_{n1} \\ \delta_{l2} & \delta_{m2} & \delta_{n2} \\ \delta_{l3} & \delta_{m3} & \delta_{n3} \end{bmatrix} \right| && (24) \\ &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} && (25)\end{aligned}$$

on expanding the determinant in the Eqn(25), we obtain

$$\begin{aligned}\epsilon_{ijk}\epsilon_{lmn} &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) \\ &\quad +\delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) && (26)\end{aligned}$$

(a) set $k = p$ and $n = p$, in the Eqn(26)

$$\begin{aligned}\epsilon_{ijp}\epsilon_{lmp} &= \delta_{il}(\delta_{jm}\delta_{pp} - \delta_{jp}\delta_{pm}) + \delta_{im}(\delta_{jp}\delta_{pl} - \delta_{jl}\delta_{pp}) \\ &\quad +\delta_{ip}(\delta_{jl}\delta_{pm} - \delta_{jm}\delta_{pl}) && (27) \\ &= 3\,\delta_{il}\delta_{jm} - \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} - 3\,\delta_{im}\delta_{jl} \\ &\quad +\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} && (28) \\ &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} && (29)\end{aligned}$$

(b) since p is a dummy index in Eqn(29) replace it by the q

$$\epsilon_{ijq}\epsilon_{lmq} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{30}$$

in the above equation now set $j = p$**, and** $m = p$**, to obtain**

$$\begin{aligned} \epsilon_{ipq}\epsilon_{lpq} &= \delta_{il}\delta_{pp} - \delta_{ip}\delta_{pl} \\ &= 3\,\delta_{il} - \delta_{il} \\ &= 2\,\delta_{il} \end{aligned} \tag{31}$$

3. Let A be an arbitrary tensor and $\mathbf{A}^*$ its adjugate. (i) Given that A_{ij} are the components of **A** relative to an orthonormal basis e, show that the components of $\mathbf{A}^*$ are $\frac{1}{2}\epsilon_{ipq}\epsilon_{jrs}A_{pr}A_{qs}$. Deduce that $A^{T*} = A^*$
(ii) Show that
(a) $(\mathbf{A}^*)^* = (\mathbf{det}\,\mathbf{A})\mathbf{A}$,
(b) tr $\mathbf{A}^* = II_A$,
(c) $\mathbf{A}\{\mathbf{a} \wedge (\mathbf{A}^{\mathbf{T}}\mathbf{b})\} = (\mathbf{A}^* \wedge \mathbf{b}) \quad \forall\, \mathbf{a},\, \mathbf{b} \in \mathbf{E}$
Soln.
(i) using the defintion of an adjugate,

$$\mathbf{A}^*(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{A}\,\mathbf{a}) \wedge (\mathbf{A}\,\mathbf{b}) \tag{32}$$

replacing **a** and **b** with the orthonormal basis e_i and e_j we get,

$$\mathbf{A}^*(\mathbf{e_i} \wedge \mathbf{e_j}) = (\mathbf{A}\,\mathbf{e_i}) \wedge (\mathbf{A}\,\mathbf{e_j}) \tag{33}$$

$$\mathbf{A^*_{pq}}\,(\mathbf{e_p} \otimes \mathbf{e_q})(\mathbf{e_i} \wedge \mathbf{e_j}) = \mathbf{A_{mn}}\,(\mathbf{e_m} \otimes \mathbf{e_n})\mathbf{e_i} \wedge \mathbf{A_{rs}}\,(\mathbf{e_r} \otimes \mathbf{e_s})\mathbf{e_j} \tag{34}$$

Using $e_i \wedge e_j = \epsilon_{kij}e_k$, and $(\mathbf{a} \otimes \mathbf{b})\,\mathbf{c} = (\mathbf{c}.\mathbf{b})\mathbf{a}$, we obtain

$$\mathbf{A^*_{pq}}\,(\mathbf{e_p} \otimes \mathbf{e_q})\epsilon_{\mathbf{kij}}\mathbf{e_k} = \mathbf{A_{mn}}\,(\mathbf{e_i} \cdot \mathbf{e_n})\mathbf{e_m} \wedge \mathbf{A_{rs}}\,(\mathbf{e_j} \cdot \mathbf{e_s})\mathbf{e_r} \tag{35}$$

Using $e_i \cdot e_j = \delta_{ij}$, we obtain

$$\mathbf{A^*_{pq}}\,(\mathbf{e_k} \otimes \mathbf{e_q})\mathbf{e_p}\epsilon_{\mathbf{kij}} = \mathbf{A_{mn}}\,\delta_{\mathbf{in}}\mathbf{e_m} \wedge \mathbf{A_{rs}}\,\delta_{\mathbf{js}}\mathbf{e_r}$$

$$\mathbf{A^*_{pq}}\,\delta_{\mathbf{kq}}\mathbf{e_p}\epsilon_{\mathbf{kij}} = \mathbf{A_{mn}}\,\delta_{\mathbf{in}}\mathbf{e_m} \wedge \mathbf{A_{rs}}\,\delta_{\mathbf{js}}\mathbf{e_r}$$

$$\mathbf{A^*_{pq}}\,\mathbf{e_p}\epsilon_{\mathbf{qij}} = \mathbf{A_{mi}}\mathbf{A_{rj}}\,\mathbf{e_m} \wedge\, \mathbf{e_r} \tag{36}$$

$$\mathbf{A^*_{pq}}\,\mathbf{e_p}\epsilon_{\mathbf{qij}} = \mathbf{A_{mi}}\mathbf{A_{rj}}\,\epsilon_{\mathbf{smr}}\mathbf{e_s} \tag{37}$$

taking the dot product with e_k on both sides, we obtain

$$\begin{aligned} \mathbf{A^*_{pq}}\epsilon_{\mathbf{qij}}(\mathbf{e_p} \cdot \mathbf{e_k}) &= \mathbf{A_{mi}}\mathbf{A_{rj}}\epsilon_{\mathbf{smr}}(\mathbf{e_s} \cdot \mathbf{e_k}) \\ \mathbf{A^*_{pq}}\epsilon_{\mathbf{qij}}\delta_{\mathbf{pk}} &= \mathbf{A_{mi}}\mathbf{A_{rj}}\epsilon_{\mathbf{smr}}\delta_{\mathbf{sk}} \\ \mathbf{A^*_{kq}}\epsilon_{\mathbf{qij}} &= \mathbf{A_{mi}}\mathbf{A_{rj}}\epsilon_{\mathbf{kmr}} \end{aligned} \tag{38}$$

Multiplying both sides with ϵ_{sij} we obtain

$$\epsilon_{sij}\epsilon_{qij}\mathbf{A^*_{kq}} = \epsilon_{\mathbf{sij}}\epsilon_{\mathbf{kmr}}\mathbf{A_{mi}}\mathbf{A_{rj}} \tag{39}$$

using, $\epsilon_{sij}\epsilon_{qij} = 2\delta_{sq}$

$$2\,\delta_{sq}\mathbf{A^*_{kq}} = \epsilon_{\mathbf{sij}}\epsilon_{\mathbf{kmr}}\mathbf{A_{mi}}\mathbf{A_{rj}} \tag{40}$$

$$\mathbf{A^*_{ks}} = \frac{1}{2}\epsilon_{\mathbf{sij}}\epsilon_{\mathbf{kmr}}\mathbf{A_{mi}}\mathbf{A_{rj}} \tag{41}$$

which, by replacing the dummy varibles can be written as,

$$\mathbf{A^*_{ij}} = \frac{1}{2}\epsilon_{\mathbf{ipq}}\epsilon_{\mathbf{jrs}}\mathbf{A_{pr}}\mathbf{A_{qs}} \tag{42}$$

Now,

$$
\begin{aligned}
(\mathbf{A}^{\mathbf{T}})^*_{\mathbf{ij}} &= \frac{1}{2}\epsilon_{\mathbf{ipq}}\epsilon_{\mathbf{jrs}}\mathbf{A}^{\mathbf{T}}_{\mathbf{pr}}\mathbf{A}^{\mathbf{T}}_{\mathbf{qs}} \\
&= \frac{1}{2}\epsilon_{\mathbf{ipq}}\epsilon_{\mathbf{jrs}}\mathbf{A}_{\mathbf{rp}}\mathbf{A}_{\mathbf{sq}} \qquad (43)
\end{aligned}
$$

on replacing the dummy variables Eqn(43) can be rewritten as

$$
\begin{aligned}
(\mathbf{A}^{\mathbf{T}})^*_{\mathbf{ij}} &= \frac{1}{2}\epsilon_{\mathbf{irs}}\epsilon_{\mathbf{jpq}}\mathbf{A}_{\mathbf{pr}}\mathbf{A}_{\mathbf{qs}} \\
&= \mathbf{A}^*_{\mathbf{ji}} \\
&= (\mathbf{A}^{*^{\mathbf{T}}})_{\mathbf{ij}} \qquad (44)
\end{aligned}
$$

Thus, $(\mathbf{A}^{\mathbf{T}})^* = \mathbf{A}^{*^{\mathbf{T}}}$

(ii) (a) consider the case when $\mathbf{det\, A} \neq 0$,
In that case there exists an inverse A^{-1} and we know that

$$(A^T)^* = (A^*)^T = \frac{A^{-1}}{(det A)^{-1}} = (det\, A)\, A^{-1}$$

or

$$A^* = (det\, A)\, A^{-T} = B$$

We need to find $(A^*)^*$,

$$\mathbf{det(A^{-T}) = det(A^{-1}) = \frac{1}{det\, A}} \tag{45}$$

$$\mathbf{det\, B = det(det\, A\, A^{-T}) = (det\, A)^3 det(A^{-T}) = (det\, A)^2} \tag{46}$$

now,

$$\begin{aligned} \mathbf{B^{-T} = (A^*)^{-T}} &= \mathbf{(A^{-T} det\, A)^{-T}} \\ &= \mathbf{\frac{1}{det\, A}(A^{-T})^{-T}} \\ &= \mathbf{\frac{A}{det\, A}} \end{aligned} \tag{47}$$

$$\mathbf{(A^*)^* = B^* = (det\, B)\, B^{-T}} \tag{48}$$

on substuting Eqns(46 and 47) in Eqn(48) we obtain

$$\mathbf{(A^*)^* = B^* = det\, A\ A} \tag{49}$$

(b)

$$\mathbf{II_A}\,[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{a},\mathbf{Ab},\mathbf{Ac}] + [\mathbf{Aa},\mathbf{b},\mathbf{Ac}] + [\mathbf{Aa},\mathbf{Ab},\mathbf{c}] \tag{50}$$

we know that $\mathbf{A}^*\,(\mathbf{a}\wedge\mathbf{b}) = \mathbf{A}\,\mathbf{a}\,\wedge\,\mathbf{A}\,\mathbf{b}$ and also,
$\mathbf{B}\,(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c} = (\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{B}^{\mathbf{T}}\mathbf{c}$

$$\begin{aligned}\mathbf{II_A}\,[\mathbf{a},\mathbf{b},\mathbf{c}] &= [(\mathbf{A}^*)^{\mathbf{T}}\,\mathbf{a},\mathbf{b},\mathbf{c}] + [\mathbf{a},(\mathbf{A}^*)^{\mathbf{T}}\,\mathbf{b},\mathbf{c}] + [\mathbf{a},\mathbf{b},(\mathbf{A}^*)^{\mathbf{T}}\,\mathbf{c}] \\ &= \mathrm{tr}\,(\mathbf{A}^*)^{\mathbf{T}}[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathrm{tr}\,\mathbf{A}^*\,[\mathbf{a},\mathbf{b},\mathbf{c}]\end{aligned} \tag{51}$$

(c) Method-1

By definition

$$(\mathbf{A}^*)^* = \mathbf{det\,A\;A} \tag{52}$$

or

$$\mathbf{A} = \frac{(\mathbf{A}^*)^*}{\mathbf{det\,A}} \tag{53}$$

and using $\mathbf{B}^*(\mathbf{a}\wedge\mathbf{b}) = (\mathbf{Ba})\wedge(\mathbf{Bb})$

$$\begin{aligned}\mathbf{A}\left\{\mathbf{a}\wedge(\mathbf{A}^{\mathrm{T}}\mathbf{b})\right\} &= \frac{(\mathbf{A}^*)^*}{\mathbf{det\,A}}\left\{\mathbf{a}\wedge(\mathbf{A}^{\mathrm{T}}\mathbf{b})\right\} \\ &= \frac{\mathbf{1}}{\mathbf{det\,A}}\left\{\mathbf{A}^*\mathbf{a}\wedge\mathbf{A}^*(\mathbf{A}^{\mathrm{T}}\mathbf{b})\right\}\end{aligned} \tag{54}$$

Using $\mathbf{A}^*\,\mathbf{A}^{\mathrm{T}} = \mathbf{det\,A}$

$$\begin{aligned}\mathbf{A}\left\{\mathbf{a}\wedge(\mathbf{A}^{\mathrm{T}}\mathbf{b})\right\} &= \frac{\mathbf{1}}{\mathbf{det\,A}}\left\{\mathbf{A}^*\mathbf{a}\wedge\mathbf{det\,A\,b}\right\} \\ &= \frac{\mathbf{det\,A}}{\mathbf{det\,A}}\left\{\mathbf{A}^*\mathbf{a}\wedge\,\mathbf{b}\right\} \\ &= \left\{\mathbf{A}^*\mathbf{a}\wedge\,\mathbf{b}\right\}\end{aligned} \tag{55}$$

Method-2

consider $\mathbf{A}\left\{\mathbf{a}\wedge(\mathbf{A}^{\mathrm{T}}\mathbf{b})\right\}$

Take dot product with an arbitrary non zero vector $\mathbf{c}$

and using $\mathbf{B\,a}\cdot\mathbf{b} = \mathbf{a}\cdot\,\mathbf{B}^{\mathrm{T}}\,\mathbf{b}$

$$\begin{aligned}\mathbf{A}\{\mathbf{a}\wedge(\mathbf{A}^{\mathrm{T}}\mathbf{b})\}\cdot\mathbf{c} &= \{\mathbf{a}\wedge(\mathbf{A}^{\mathrm{T}}\mathbf{b})\}\cdot\,\mathbf{A}^{\mathrm{T}}\,\mathbf{c} \\ &= [\mathbf{a},(\mathbf{A}^{\mathrm{T}}\mathbf{b}),(\mathbf{A}^{\mathrm{T}}\,\mathbf{c})] \\ &= \{\mathbf{a}\cdot(\mathbf{A}^*)^{\mathrm{T}}(\mathbf{b}\wedge\mathbf{c})\} \\ &= \{\mathbf{A}^*\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})\} \\ &= \{\mathbf{A}^*\mathbf{a}\wedge\mathbf{b}\}\cdot\mathbf{c}\end{aligned} \tag{56}$$