EN221 - Fall2008 - HW # 2 Solutions

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1) Let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary tensors, $\mathbf{A}^*$ and $\mathbf{B}^*$ the adjugates of $\mathbf{A}$ and $\mathbf{B}$, and α and β arbitrary scalars. Show that

$$det(\alpha\mathbf{A}+\beta\mathbf{B}) = \alpha^3 det\,\mathbf{A} + \alpha^2\beta \text{ tr } (\mathbf{B}^{\mathbf{T}}\mathbf{A}^*) + \alpha\beta^2 \text{ tr } (\mathbf{A}^{\mathbf{T}}\mathbf{B}^*) + \beta^3 det\,\mathbf{B}$$

Soln.

$$det(\alpha\mathbf{A}+\beta\mathbf{B}) = [(\alpha\mathbf{A}+\beta\mathbf{B})\mathbf{e_1}, (\alpha\mathbf{A}+\beta\mathbf{B})\mathbf{e_2}, (\alpha\mathbf{A}+\beta\mathbf{B})\mathbf{e_3}] \qquad (1)$$

$$\begin{aligned} det(\alpha\mathbf{A}+\beta\mathbf{B}) &= (\alpha\mathbf{A}+\beta\mathbf{B})\mathbf{e_1}\cdot[(\alpha\mathbf{A}+\beta\mathbf{B})\mathbf{e_2}\wedge(\alpha\mathbf{A}+\beta\mathbf{B})\mathbf{e_3}] \\ &= (\alpha\mathbf{A}+\beta\mathbf{B})\mathbf{e_1}\cdot[\alpha^{\mathbf{2}}\mathbf{Ae_2}\wedge\mathbf{Ae_3}+\alpha\beta\mathbf{Be_2}\wedge\mathbf{Ae_3}\alpha\beta\mathbf{Ae_2}\wedge\mathbf{Be_3}+\beta^{\mathbf{2}}\mathbf{Be_2}\wedge\mathbf{Be_3}] \\ &= \alpha^{\mathbf{3}}\mathbf{Ae_1}\cdot\mathbf{Ae_2}\wedge\mathbf{Ae_3}+... \\ &\quad +\alpha^{\mathbf{2}}\beta[\mathbf{Be_1}\cdot\mathbf{Ae_2}\wedge\mathbf{Ae_3}+\mathbf{Ae_1}\cdot\mathbf{Ae_2}\wedge\mathbf{Be_3}+\mathbf{Ae_1}\cdot\mathbf{Be_2}\wedge\mathbf{Ae_3}]+... \\ &\quad +\alpha\beta^{\mathbf{2}}[\mathbf{Ae_1}\cdot\mathbf{Be_2}\wedge\mathbf{Be_3}+\mathbf{Be_1}\cdot\mathbf{Be_2}\wedge\mathbf{Ae_3}+\mathbf{Ae_1}\cdot\mathbf{Be_2}\wedge\mathbf{Be_3}]+... \\ &\quad +\beta^{\mathbf{3}}\mathbf{Be_1}\cdot\mathbf{Be_2}\wedge\mathbf{Be_3} \end{aligned}$$

Note that, $\mathbf{Ae_1}\cdot\mathbf{Ae_2}\wedge\mathbf{Ae_3} = det\,\mathbf{A}$ and, $\mathbf{Be_1}\cdot\mathbf{Be_2}\wedge\mathbf{Be_3} = det\,\mathbf{B}$
Now gather the terms that go with $\alpha^2\beta$

$$\mathbf{Be_1}\cdot\mathbf{Ae_2}\wedge\mathbf{Ae_3} = \mathbf{Be_1}\cdot\mathbf{A}^*(\mathbf{e_2}\wedge\mathbf{e_3}) = [\mathbf{A}^{*\mathbf{T}}\mathbf{Be_1},\mathbf{e_2},\mathbf{e_3}]$$

$$\mathbf{Ae_1}\cdot\mathbf{Ae_2}\wedge\mathbf{Be_3} = [\mathbf{e_1},\mathbf{e_2},\mathbf{A}^{*\mathbf{T}}\mathbf{Be_3}]$$

$$\mathbf{Ae_1}\cdot\mathbf{Be_2}\wedge\mathbf{Ae_3} = [\mathbf{e_1},\mathbf{A}^{*\mathbf{T}}\mathbf{Be_2},\mathbf{e_3}]$$

And the sum of the above terms are,

$$[\mathbf{A}^{*\mathbf{T}}\mathbf{Be_1},\mathbf{e_2},\mathbf{e_3}]+[\mathbf{e_1},\mathbf{e_2},\mathbf{A}^{*\mathbf{T}}\mathbf{Be_3}]+[\mathbf{e_1},\mathbf{A}^{*\mathbf{T}}\mathbf{Be_2},\mathbf{e_3}] = \text{tr } (\mathbf{A}^{*\mathbf{T}}\mathbf{B}) = \text{tr } (\mathbf{A}^{*\mathbf{T}}\mathbf{B})^{\mathbf{T}} = \text{tr } (\mathbf{B}^{\mathbf{T}}\mathbf{A}^*)$$

It can similarly be easily seen that the terms that go with $\alpha\beta^2$ is tr $(\mathbf{A}^{\mathbf{T}}\mathbf{B}^*)$
hence,

$$det(\alpha\mathbf{A}+\beta\mathbf{B}) = \alpha^3 det\,\mathbf{A} + \alpha^2\beta \text{ tr } (\mathbf{B}^{\mathbf{T}}\mathbf{A}^*) + \alpha\beta^2 \text{ tr } (\mathbf{A}^{\mathbf{T}}\mathbf{B}^*) + \beta^3 det\,\mathbf{B} \qquad (2)$$

2) Show that

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

is an improper orthogonal matrix that represents a change of basis equivalent to a reflection in the plane through $\mathbf{e_3}$ inclined at a positive angle $\mathbf{e_1}$.

Soln.
The determinant of $\mathbf{Q}$ is -1 hence it is improper orthogonal matrix .
Consider an arbitrary vector $\mathbf{a}$, $\mathbf{a} = \alpha_\mathbf{1}\mathbf{e_1} + \alpha_\mathbf{2}\mathbf{e_2} + \alpha_\mathbf{3}\mathbf{e_3}$

$$\mathbf{a} = \left\{ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \right\}$$

The components of the vector are then split into two components. One component is the plane formed by $\mathbf{e_1^{'}}$ and $\mathbf{e_3}$. Here $\mathbf{e_1^{'}}$ is the vector in the plane of $\mathbf{e_1} - \mathbf{e_2}$ and at an angle θ w.r.t. $\mathbf{e_1}$

$$\mathbf{e_1^{'}} = \cos\theta\,\mathbf{e_1} + \sin\theta\,\mathbf{e_2}$$

The vector perpendicular to the plane $\mathbf{e_1^{'}} - \mathbf{e_3}$ is

$$\begin{aligned} \mathbf{e}_\perp &= \mathbf{e_3} \wedge \mathbf{e_1^{'}} \\ &= \mathbf{e_3} \wedge (\cos\theta\,\mathbf{e_1} + \sin\theta\,\mathbf{e_2}) \\ &= \cos\theta\,\mathbf{e_2} - \sin\theta\,\mathbf{e_1} \end{aligned} \tag{4}$$

The component of $\mathbf{a}$ out of the plane $\mathbf{e_1^{'}} - \mathbf{e_3}$ is

$$\begin{aligned} \mathbf{a}_\perp &= a \cdot e_\perp \\ &= (\alpha_\mathbf{1}\,\mathbf{e_1} + \alpha_\mathbf{2}\,\mathbf{e_2} + \alpha_\mathbf{3}\,\mathbf{e_3}) \cdot (\cos\theta\,\mathbf{e_2} - \sin\theta\,\mathbf{e_1}) \\ &= -\alpha_\mathbf{1}\sin\theta + \alpha_\mathbf{2}\cos\theta \end{aligned} \tag{5}$$

and the component in the plane is

$$\begin{aligned} \mathbf{a}^{'} &= \mathbf{a} - \mathbf{a}_\perp \mathbf{e}_\perp \\ &= (\alpha_\mathbf{1}\,\mathbf{e_1} + \alpha_\mathbf{2}\,\mathbf{e_2} + \alpha_\mathbf{3}\,\mathbf{e_3}) - [-\alpha_\mathbf{1}\sin\theta + \alpha_\mathbf{2}\cos\theta][\cos\theta\,\mathbf{e_2} - \sin\theta\,\mathbf{e_1}] \\ &= [\alpha_\mathbf{1} - \alpha_\mathbf{1}\sin^\mathbf{2}\theta + \alpha_\mathbf{2}\sin\theta\cos\theta]\mathbf{e_1} + ... \\ &\quad +[\alpha_\mathbf{2} - \alpha_\mathbf{2}\cos^\mathbf{2}\theta + \alpha_\mathbf{1}\sin\theta\cos\theta]\mathbf{e_2} + ... \\ &\quad +\alpha_\mathbf{3}\mathbf{e_3} \\ &= [\alpha_\mathbf{1}\cos^\mathbf{2}\theta + \alpha_\mathbf{2}\sin\theta\cos\theta]\mathbf{e_1} + [\alpha_\mathbf{2}\sin^\mathbf{2}\theta + \alpha_\mathbf{1}\sin\theta\cos\theta]\mathbf{e_2} + \alpha_\mathbf{3}\,\mathbf{e_3} \end{aligned} \tag{6}$$

The transformation $\mathbf{Q}$ converts $\mathbf{a}$ to $\mathbf{a_r}$

$$\begin{aligned} \mathbf{a_r} &= \mathbf{Q\,a} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \\ &= \begin{Bmatrix} \alpha_1 \cos 2\theta + \alpha_2 \sin 2\theta \\ \alpha_1 \sin 2\theta - \alpha_2 \cos 2\theta \\ \alpha_3 \end{Bmatrix} \end{aligned}$$

The component of $\mathbf{a_r}$ out of plane $\mathbf{e_1^{'}} - \mathbf{e_3}$ is

$$\begin{aligned} \mathbf{a_{r\perp}} &= \mathbf{a_r} \cdot \mathbf{e_\perp} \\ &= (\alpha_\mathbf{1} \cos \mathbf{2}\theta + \alpha_\mathbf{2} \sin \mathbf{2}\theta, \alpha_\mathbf{1} \sin \mathbf{2}\theta - \alpha_\mathbf{2} \cos \mathbf{2}\theta, \alpha_\mathbf{3}) \cdot (-\sin\theta, \cos\theta, \mathbf{0}) \\ &= \alpha_\mathbf{1} \sin\theta - \alpha_\mathbf{2} \cos\theta \end{aligned} \tag{7}$$

Similarly,
the component of $\mathbf{a_r}$ in the plane $\mathbf{e_1^{'}} - \mathbf{e_3}$ is

$$\begin{aligned} \mathbf{a_r^{'}} &= \mathbf{a_r} - (\mathbf{a_r} \cdot \mathbf{e_\perp})\mathbf{e_\perp} \\ &= (\alpha_\mathbf{1} \cos^\mathbf{2}\theta + \alpha_\mathbf{2} \cos\theta \sin\theta)\mathbf{e_1} + (\alpha_\mathbf{1} \cos\theta \sin\theta + \alpha_\mathbf{2} \sin^\mathbf{2}\theta)\mathbf{e_2} + \mathbf{e_3} \end{aligned} \tag{8}$$

Eqns. 5,6,7,8 tell that

$\mathbf{a_r^{'}} = \mathbf{a}^{'}$, In plane components are equal for $\mathbf{a}$ and $\mathbf{a}^{'}$

$\mathbf{a_{r\perp}} = -\mathbf{a_\perp}$, Out of plane components are equal and opposite

But, that precisely is the definition of reflection in 3-D.

3) Consider the following symmetric matrix, **S**:

$$\begin{pmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{pmatrix} \tag{9}$$

(a) Show that **S** is positive definite. (b) Calculate **A**, the square root **S**, Verify that $\mathbf{A^2} = \mathbf{S}$.
Soln.
The eigenvalues of S are obtained from

$$\begin{vmatrix} 5-\lambda & 0 & 4 \\ 0 & 4-\lambda & 0 \\ 4 & 0 & 4-\lambda \end{vmatrix} = 0 \tag{10}$$

$$\begin{aligned} &(5-\lambda)[(4-\lambda)(5-\lambda)] + 4[0 - 4(4-\lambda)] = 0 \\ &(4-\lambda)[(5-\lambda)^2 - 16] = 0 \\ &(4-\lambda)[(5-\lambda-4)(5-\lambda+4)] = 0 \\ &(4-\lambda)(1-\lambda)(9-\lambda) = 0 \end{aligned} \tag{11}$$

the three roots are, $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 9$
Since all the eigenvalues are positive the matrix **S** is +ve definite

To find the eigenvectors

$$\begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = 1 \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{12}$$

on solving we obtain $\alpha_1 = \alpha_3$ and $\alpha_2 = 0$
so the unit eigenvector is,

$$p_1 = \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{Bmatrix} \tag{13}$$

$$\begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = 4 \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{14}$$

so the unit eigenvector is,

$$p_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \tag{15}$$

$$\begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = 9 \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{16}$$

so the unit eigenvector is,

$$p_3 = \left\{ \begin{array}{c} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{array} \right\} \tag{17}$$

The square root matrix $\mathbf{S}^{1/2}$ is written as

$$\begin{aligned} \mathbf{A} = \mathbf{S}^{1/2} &= \sum_{i=1}^{3} \lambda_i^{1/2}\, p_i \otimes p_i \\ &= \sqrt{1}\, p_1 \otimes p_1 + \sqrt{4}\, p_2 \otimes p_2 + \sqrt{9}\, p_3 \otimes p_3 \\ &= 1 \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \end{aligned} \tag{18}$$

Checking

$$A * A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix} = \mathbf{S} \tag{19}$$

4) if det $\mathbf{T} \neq 0$ deduce using det $(\mathbf{T} - \lambda\mathbf{I}) = 0$ that

$$\det\,(\mathbf{T^{-1}} - \lambda^{-1}\mathbf{I}) = \mathbf{0}$$

and hence

$$\lambda^{-\mathbf{3}} - \mathbf{I_1}(\mathbf{T^{-1}})\lambda^{-\mathbf{2}} + \mathbf{I_2}(\mathbf{T^{-1}})\lambda^{-\mathbf{1}} - \mathbf{I_3}(\mathbf{T^{-1}}) = \mathbf{0}$$

show that

$$\mathbf{I_1}(\mathbf{T^{-1}}) = \mathbf{I_2}(\mathbf{T})/\mathbf{I_3}(\mathbf{T})$$
$$\mathbf{I_2}(\mathbf{T^{-1}}) = \mathbf{I_1}(\mathbf{T})/\mathbf{I_3}(\mathbf{T})$$
$$\mathbf{I_3}(\mathbf{T^{-1}}) = \mathbf{1}/\mathbf{I_3}(\mathbf{T})$$

Soln.

Since det$\mathbf{T} \neq 0$, $\mathbf{T^{-1}}$ exists, and using $\mathbf{T}\,\mathbf{T^{-1}} = \mathbf{I}$

$$\begin{aligned} 0 = \det(\mathbf{T} - \lambda\mathbf{I}) &= \det(\mathbf{T} - \lambda\mathbf{T}\,\mathbf{T^{-1}}) \\ &= \det(-\lambda[\mathbf{T}\,\mathbf{T^{-1}} - \mathbf{T}\lambda^{-\mathbf{1}}]) \\ &= \det(-\lambda\mathbf{T}\,[\mathbf{T^{-1}} - \mathbf{I}\lambda^{-\mathbf{1}}]) \\ &= -\lambda^3\det(\mathbf{T})\,\det(\mathbf{T^{-1}} - \mathbf{I}\lambda^{-\mathbf{1}}) \\ &= -\lambda^3\,\mathbf{I_3}(\mathbf{T})\,\det(\mathbf{T^{-1}} - \mathbf{I}\lambda^{-\mathbf{1}}) \end{aligned} \tag{20}$$

since det$\mathbf{T} \neq 0$, so $\lambda \neq 0$
hence, Eqn.(20) can be written as det $(\mathbf{T^{-1}} - \lambda^{-1}\mathbf{I}) = \mathbf{0}$

using Eqn.(20) and expanding the determinants on both sides

$$\begin{aligned} \det(\mathbf{T} - \lambda\mathbf{I}) &= -\lambda^3\,\mathbf{I_3}(\mathbf{T})\,\det(\mathbf{T^{-1}} - \mathbf{I}\lambda^{-\mathbf{1}}) \\ \lambda^{\mathbf{3}} - \mathbf{I_1}(\mathbf{T})\lambda^{\mathbf{2}} + \mathbf{I_2}(\mathbf{T})\lambda - \mathbf{I_3}(\mathbf{T}) &= -\lambda^{\mathbf{3}}\,\mathbf{I_3}(\mathbf{T})\,[\lambda^{-\mathbf{3}} - \mathbf{I_1}(\mathbf{T^{-1}})\lambda^{-\mathbf{2}} + \mathbf{I_2}(\mathbf{T^{-1}})\lambda^{-\mathbf{1}} - \mathbf{I_3}(\mathbf{T^{-1}})] \\ 0 = \frac{\mathbf{1}}{\mathbf{I_3}(\mathbf{T})}\lambda^{\mathbf{3}} - \frac{\mathbf{I_1}(\mathbf{T})}{\mathbf{I_3}(\mathbf{T})}\lambda^{\mathbf{2}} + \frac{\mathbf{I_2}(\mathbf{T})}{\mathbf{I_3}(\mathbf{T})}\lambda - \mathbf{1} &= -\mathbf{1} + \mathbf{I_1}(\mathbf{T^{-1}})\lambda - \mathbf{I_2}(\mathbf{T^{-1}})\lambda^{\mathbf{2}} + \mathbf{I_3}(\mathbf{T^{-1}})\lambda^{\mathbf{3}} = 0 \end{aligned} \tag{21}$$

on comparing the coefficients of λ, λ^2 and λ^3 respectively, we get

$$\mathbf{I_1}(\mathbf{T^{-1}}) = \mathbf{I_2}(\mathbf{T})/\mathbf{I_3}(\mathbf{T}) \tag{22}$$
$$\mathbf{I_2}(\mathbf{T^{-1}}) = \mathbf{I_1}(\mathbf{T})/\mathbf{I_3}(\mathbf{T}) \tag{23}$$
$$\mathbf{I_3}(\mathbf{T^{-1}}) = \mathbf{1}/\mathbf{I_3}(\mathbf{T}) \tag{24}$$