EN221 - Fall2008 - HW # 3 Solutions

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1). Let $\mathbf{W^{(1)}}$ and $\mathbf{W^{(2)}}$ be skew-symmetric tensors with axial vectors $\mathbf{w^{(1)}}$ and $\mathbf{w^{(2)}}$ respectively. Show that

(a) $\mathbf{W^{(1)}W^{(2)} = w^{(1)} \otimes w^{(2)} - (w^{(1)} \cdot w^{(2)})I}$
(b) tr $(\mathbf{W^{(1)}W^{(2)}}) = \mathbf{-2\, w^{(1)} \cdot w^{(2)}}$

By making appropriate choices of $\mathbf{W^{(1)}}$ and $\mathbf{W^{(2)}}$, deduce from (b) the vector identity

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \quad \forall\, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \in \mathbf{E}.$$

Soln.
(a)
Using the relation between the Skew-Symmetric matrix and its axial vector from Ogden Pg. 24 (One can easily show this by writing $\mathbf{W\,a = w \wedge a}$ in indicial notation)

$$\mathbf{W^{(1)}_{ij}} = -\epsilon_{\mathbf{ijk}}\mathbf{w^{(1)}_k} \tag{1}$$
$$\mathbf{W^{(2)}_{jl}} = -\epsilon_{\mathbf{jlr}}\mathbf{w^{(2)}_r} \tag{2}$$
$$\tag{3}$$

$$\begin{aligned}
\mathbf{W^{(1)}_{ij}W^{(2)}_{jl}} &= (-\epsilon_{\mathbf{ijk}}\mathbf{w^{(1)}_k})(-\epsilon_{\mathbf{jlr}}\mathbf{w^{(2)}_r}) \\
&= (\epsilon_{\mathbf{ijk}}\epsilon_{\mathbf{jlr}})\mathbf{w^{(1)}_k}\,\mathbf{w^{(2)}_r} \\
&= -(\epsilon_{\mathbf{jik}}\epsilon_{\mathbf{jlr}})\mathbf{w^{(1)}_k}\,\mathbf{w^{(2)}_r} \\
&= -(\delta_{\mathbf{il}}\delta_{\mathbf{kr}} - \delta_{\mathbf{ir}}\delta_{\mathbf{kl}})\mathbf{w^{(1)}_k}\,\mathbf{w^{(2)}_r} \\
&= (\delta_{\mathbf{ir}}\delta_{\mathbf{kl}} - \delta_{\mathbf{il}}\delta_{\mathbf{kr}})\mathbf{w^{(1)}_k}\,\mathbf{w^{(2)}_r} \\
&= \mathbf{w^{(1)}_l}\,\mathbf{w^{(2)}_i} - (\mathbf{w^{(1)}_k}\,\mathbf{w^{(2)}_k})\delta_{\mathbf{il}} \\
&= (\mathbf{w^{(2)}_i}\,\mathbf{w^{(1)}_l} - (\mathbf{w^{(1)}_k}\,\mathbf{w^{(2)}_k})\delta_{\mathbf{il}} \\
(\mathbf{W^{(1)}W^{(2)}})_{\mathbf{il}} &= (\mathbf{w^{(2)} \otimes w^{(1)}})_{\mathbf{il}} - (\mathbf{w^{(1)} \cdot w^{(2)}})\delta_{\mathbf{il}} \qquad (4) \\
\mathbf{W^{(1)}W^{(2)}} &= \mathbf{w^{(2)} \otimes w^{(1)} - (w^{(1)} \cdot w^{(2)})I} \qquad (5)
\end{aligned}$$

(b)

$$\text{tr}\,(\mathbf{W^{(1)}W^{(2)}}) = \text{tr}\,(\mathbf{w^{(2)}}\otimes\mathbf{w^{(1)}}) - (\mathbf{w^{(1)}}\cdot\mathbf{w^{(2)}})\,\text{tr}\,\mathbf{I} \tag{6}$$

Using the fact tr $\mathbf{a}\otimes\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$ and tr $\mathbf{I} = 3$

$$\begin{aligned}\text{tr}\,(\mathbf{W^{(1)}W^{(2)}}) &= (\mathbf{w^{(2)}}\cdot\mathbf{w^{(1)}}) - \mathbf{3}(\mathbf{w^{(1)}}\cdot\mathbf{w^{(2)}}) \\ &= -2\,(\mathbf{w^{(2)}}\cdot\mathbf{w^{(1)}})\end{aligned} \tag{7}$$

(c)
Now choose
$\mathbf{W^{(1)}} = (\mathbf{a}\otimes\mathbf{b}) - (\mathbf{b}\otimes\mathbf{a})$ and
$\mathbf{W^{(2)}} = (\mathbf{c}\otimes\mathbf{d}) - (\mathbf{d}\otimes\mathbf{c})$
Then the axial vector are $\mathbf{b}\wedge\mathbf{a}$ and $\mathbf{d}\wedge\mathbf{c}$ respectively(Chadwick pg. 30)

$$\begin{aligned}\mathbf{W^{(1)}_{ij}W^{(2)}_{jl}} &= (\mathbf{a}\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{a})\cdot(\mathbf{c}\otimes\mathbf{d} - \mathbf{d}\otimes\mathbf{c}) \\ &= (\mathbf{a}\otimes\mathbf{b}).(\mathbf{c}\otimes\mathbf{d}) - (\mathbf{a}\otimes\mathbf{b}).(\mathbf{d}\otimes\mathbf{c}) - (\mathbf{b}\otimes\mathbf{a}).(\mathbf{c}\otimes\mathbf{d}) + (\mathbf{b}\otimes\mathbf{a}).(\mathbf{d}\otimes\mathbf{c}) \\ &= (\mathbf{a}\otimes\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) - (\mathbf{a}\otimes\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\otimes\mathbf{d})(\mathbf{a}\cdot\mathbf{c}) + (\mathbf{b}\otimes\mathbf{c})(\mathbf{a}\cdot\mathbf{d})\end{aligned} \tag{8}$$

Now taking the trace of the above equation

$$\text{tr}\,(\mathbf{W^{(1)}W^{(2)}}) = \text{tr}\,((\mathbf{a}\otimes\mathbf{d})(\mathbf{b}\cdot\mathbf{c})) - \text{tr}\,((\mathbf{a}\otimes\mathbf{c})(\mathbf{b}\cdot\mathbf{d})) - \text{tr}\,((\mathbf{b}\otimes\mathbf{d})(\mathbf{a}\cdot\mathbf{c})) + \text{tr}\,((\mathbf{b}\otimes\mathbf{c})(\mathbf{a}\cdot\mathbf{d})) \tag{9}$$

Using the fact tr $(\alpha\mathbf{A}) = \alpha\,\text{tr}\,\mathbf{A}$ and tr $(\mathbf{u}\otimes\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}$

$$\begin{aligned}\text{tr}\,(\mathbf{W^{(1)}W^{(2)}}) &= (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{c}) + (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) \\ &= \mathbf{2}(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) - \mathbf{2}(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})\end{aligned} \tag{10}$$

Now in the Eqn(7) Substitute the axial vectors $\mathbf{w^{(1)}} = \mathbf{b}\wedge\mathbf{a}$ and $\mathbf{w^{(2)}} = \mathbf{d}\wedge\mathbf{c}$

$$\begin{aligned}\text{tr}\,(\mathbf{W^{(1)}W^{(2)}}) &= -\mathbf{2}(\mathbf{b}\wedge\mathbf{a})\cdot(\mathbf{d}\wedge\mathbf{c}) \\ &= -\mathbf{2}(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d})\end{aligned} \tag{11}$$

LHS of Eqns(10 and 11) are one and the same, hence on equating the RHS of both the equation we get

$$(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{12}$$

2). Let ϕ **u** and **T** be differentiable scalar, vector and tensor fields. Show that
(i) grad(ϕ**u**)= **u**$\otimes$ grad ϕ +ϕ grad **u**
(ii) div(**Tu**)= **u**$\cdot$ div **T** +tr (**T** grad **u**)
(iii) grad(ϕ**T**)= $\mathbf{T}^{\mathrm{T}}$ grad ϕ +ϕ div **T**
Soln.
(i) Let $\mathbf{v} = \phi\mathbf{u}$

$$\begin{aligned}
\mathrm{grad}\,\mathbf{v} &= \frac{\partial v_p}{\partial x_q}\, e_p \otimes e_q \\
&= \left[\frac{\partial(\phi\mathbf{u})_p}{\partial x_q}\right]\, e_p \otimes e_q \\
&= \left[\phi\frac{\partial u_p}{\partial x_q} + \frac{\partial \phi}{\partial x_q}\, u_p\right]\, e_p \otimes e_q \\
&= \left[\phi\frac{\partial u_p}{\partial x_q}\, e_p \otimes e_q + \frac{\partial \phi}{\partial x_q}\, u_p\, e_p \otimes e_q\right] \\
&= \phi\,\mathrm{grad}\,\mathbf{u} + \mathbf{u} \otimes\, \mathrm{grad}\,\phi
\end{aligned} \tag{13}$$

(ii) Let $\mathbf{v} = \mathbf{T}\,\mathbf{u}$

$$\begin{aligned}
\mathrm{div}\,\mathbf{v} &= \frac{\partial v_p}{\partial x_p} \\
&= \frac{\partial(T_{pi} u_i)}{\partial x_p} \\
&= \frac{\partial T_{pi}}{\partial x_p} u_i + T_{pi}\frac{\partial u_i}{\partial x_p} \\
&= \mathbf{u} \cdot \mathrm{div}\,\mathbf{T} + \mathrm{tr}\,(\mathbf{T}\,\mathrm{grad}\,\mathbf{u})
\end{aligned} \tag{14}$$

(iii) Let $\mathbf{P} = \phi\mathbf{T}$

$$\begin{aligned}
\mathrm{div}\,\mathbf{P} &= \frac{\partial P_{ij}}{\partial x_i} e_j \\
&= \frac{\partial(\phi\mathbf{T})_{ij}}{\partial x_i} e_j \\
&= \left[\phi\frac{\partial T_{ij}}{\partial x_i} + T_{ij}\frac{\partial \phi}{\partial x_i}\right] e_j \\
&= \left[\phi\frac{\partial T_{ij}}{\partial x_i} + T^{T}_{ji}\frac{\partial \phi}{\partial x_i}\right] e_j \\
&= \phi\,\mathrm{div}\,\mathbf{T} + \mathbf{T}^{\mathrm{T}}\,\mathrm{grad}\,\phi
\end{aligned} \tag{15}$$

3). A Pure strain is defined by the deformation gradient

$$\mathbf{A} = \lambda_1\, \mathrm{e}_1 \otimes \mathrm{e}_1 + \lambda_2\, \mathrm{e}_2 \otimes \mathrm{e}_2 + \mathrm{e}_3 \otimes \mathrm{e}_3,$$

where e_1, e_2, e_3 are mutually orthogonal unit vectors. If two line elements are aligned with $\mathbf{M} = \cos\phi\, \mathrm{e}_1 + \sin\phi\, \mathrm{e}_2$ and $\mathbf{M}^{'} = -\sin\phi\, \mathrm{e}_1 + \cos\phi\, \mathrm{e}_2$ in the reference configuration, calculate the change in the angle between them due to the deformation. Deduce that the maximum change among all such pairs of line elements is

$$\sin^{-1}\left\{\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2}\right\}$$

Soln.
Strain is defined as

$$\mathbf{A} = \lambda_1\, \mathrm{e}_1 \otimes \mathrm{e}_1 + \lambda_2\, \mathrm{e}_2 \otimes \mathrm{e}_2 + \mathrm{e}_3 \otimes \mathrm{e}_3 \tag{16}$$

The matrix representation of $\mathbf{A}$ in the basis of e_1, e_2, e_3 is

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

$$\mathbf{M} = \cos\phi\, \mathbf{e_1} + \sin\phi \mathbf{e_2}; \qquad \mathbf{M} = \left\{ \begin{array}{c} \cos\phi \\ \sin\phi \\ 0 \end{array} \right\} \tag{18}$$

$$\mathbf{M}^{'} = -\sin\phi\, \mathbf{e_1} + \cos\phi \mathbf{e_2}; \qquad \mathbf{M}^{'} = \left\{ \begin{array}{c} -\sin\phi \\ \cos\phi \\ 0 \end{array} \right\} \tag{19}$$

$$\tag{20}$$

angle between $\mathbf{M}$ and $\mathbf{M}^{'}$ is

$$\cos\Theta = \mathbf{M} \cdot \mathbf{M}^{'} = \mathbf{0} \tag{21}$$

so, $\Theta = (n + \frac{1}{2})\pi$; $\Theta = \frac{pi}{2}, \frac{3\pi}{2}$ but since $\mathbf{M}^{'} = -\sin\phi\, \mathrm{e}_1 + \cos\phi\, \mathrm{e}_2$, as opposed to, $\sin\phi\, e_1 - \cos\phi\, e_2$, hence $\Theta = \frac{pi}{2}$

After deformation

$$\mathbf{m} = \mathbf{A}\,\mathbf{M} = \left\{ \begin{array}{c} \lambda_1 \cos\phi \\ \lambda_2 \sin\phi \\ 0 \end{array} \right\} \tag{22}$$

$$\mathbf{m}^{'} = \mathbf{A}\,\mathbf{M}^{'} = \left\{ \begin{array}{c} -\lambda_1 \sin\phi \\ \lambda_2 \cos\phi \\ 0 \end{array} \right\} \tag{23}$$

so, unit vectors along $\mathbf{m}^{'}$ and $\mathbf{m}$ are

$$\mathbf{m_1} = \frac{1}{\sqrt{\lambda_1^2 \sin^2\phi + \lambda_2^2 \cos^2\phi}} \langle \lambda_1 \cos\phi, \lambda_2 \sin\phi, 0 \rangle^T \tag{24}$$

$$\mathbf{m_1^{'}} = \frac{1}{\sqrt{\lambda_1^2 \cos^2\phi + \lambda_2^2 \sin^2\phi}} \langle -\lambda_1 \sin\phi, \lambda_2 \cos\phi, 0 \rangle^T \tag{25}$$

so,

$$\cos\theta = \mathbf{m_1} \cdot \mathbf{m_1^{'}} = \frac{(\boldsymbol{\lambda_2^2} - \boldsymbol{\lambda_1^2})\cos\phi\sin\phi}{\sqrt{(\boldsymbol{\lambda_1^2}\sin^{\mathbf{2}}\phi + \boldsymbol{\lambda_2^2}\cos^{\mathbf{2}}\phi)(\boldsymbol{\lambda_1^2}\cos^{\mathbf{2}}\phi + \boldsymbol{\lambda_2^2}\sin^{\mathbf{2}}\phi)}} \tag{26}$$

Let $\alpha = \Theta - \theta = \frac{\pi}{2} - \theta$
we are looking for maximum α
take $\sin\alpha = \cos\theta$
hence for maximum α we have to find maximum $\cos\theta$
If one looks at the Eqn(26) and interchanges $\cos\phi$ by $\sin\phi$ the Eqn. remains unchanged hence the maximum of Eqn(26) is obtained by solving the maximum of either the numerator or denominator in the LHS.

The maximum of numerator is easily obtained to be when $\cos\phi = \sin\phi$,
Hence $\phi = \frac{\pi}{4}, \frac{3\pi}{4}$
so on substituting either $\phi = \frac{\pi}{4}, \frac{3\pi}{4}$ in Eqn(26) we obtain

$$\sin\alpha_{max} = \frac{|\lambda_1^2 - \lambda_2^2|}{\lambda_1^2 + \lambda_2^2} \tag{27}$$

or,

$$\alpha_{max} = \sin^{-1}\left(\frac{|\lambda_1^2 - \lambda_2^2|}{\lambda_1^2 + \lambda_2^2}\right) \tag{28}$$