EN221 - Fall2008 - HW # 4 Solutions

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1). From Nanson's formula deduce that

$$(\mathrm{ds/dS})^2 = \mathrm{J}^2 \mathbf{N} \cdot (\mathbf{B}^{\mathbf{T}} \mathbf{B}\, \mathbf{N})$$

For the simple shear deformation

$$\mathbf{A} = \mathbf{I} + \gamma \mathrm{e}_1 \otimes \mathrm{e}_2$$

Show that the inverse deformation is given by

$$\mathbf{B} = \mathbf{I} - \gamma \mathrm{e}_2 \otimes \mathrm{e}_1$$

Hence find $\mathbf{B}^{\mathbf{T}}\mathbf{B}$, and show that the maximum value of $(\mathrm{ds/dS})^2$ is

$$1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\gamma(4+\gamma^2)^{\frac{1}{2}}$$

and corresponds to N_3 and $N_1^2 - N_2^2 = -2\gamma\, N_1 N_2$. what is the minimum value of $(\mathrm{ds/dS})^2$?
Soln.

$$\begin{aligned} d\overline{s} &= J\, B\, d\overline{S} \\ \mathbf{n}\mathrm{ds} &= J\, \mathbf{B}\, \mathbf{N}\mathrm{dS} \end{aligned} \qquad (1)$$

on taking dot product with same

$$\begin{aligned} \mathbf{n}.\mathbf{n}(\mathrm{ds})^2 &= J^2\, \mathbf{B}\, \mathbf{N} \cdot \mathbf{B}\, \mathbf{N}(\mathrm{dS})^2 \\ \left(\frac{ds}{dS}\right)^2 &= J^2\, \mathbf{N} \cdot \mathbf{B}^{\mathbf{T}} \mathbf{B}\, \mathbf{N} \end{aligned} \qquad (2)$$

consider the simple sheat deformation

$$\mathbf{A} = \mathbf{I} + \gamma\, e_1 \otimes e_2 \qquad (3)$$

$$J = \det \mathbf{A} = 1 \qquad (4)$$

The inverse of the deformation gradient (using mathematica)

$$\mathbf{A}^{-\mathbf{1}} = \mathbf{I} - \gamma\, e_1 \otimes e_2 \qquad (5)$$

Therefore

$$\mathbf{B} = \mathbf{A}^{-\mathbf{T}} = \mathbf{I} - \gamma\, e_2 \otimes e_1 \tag{6}$$

$$\mathbf{B}^{\mathbf{T}}\mathbf{B} = \begin{bmatrix} 1+\gamma^2 & -\gamma & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

The eigenvalues of $\mathbf{B}^{\mathbf{T}}\mathbf{B}$ are
$\lambda_1 = 1, \quad \lambda_2 = \frac{1}{2}(2+\gamma^2-\gamma\sqrt{4+\gamma^2}), \quad \lambda_3 = \frac{1}{2}(2+\gamma^2+\gamma\sqrt{4+\gamma^2})$ the corresponding eigen-vectors

$$\left\{\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right\}, \quad \left\{\begin{array}{c} \frac{1}{2}(-\gamma+\sqrt{4+\gamma^2}) \\ 1 \\ 0 \end{array}\right\}, \quad \left\{\begin{array}{c} \frac{1}{2}(-\gamma-\sqrt{4+\gamma^2}) \\ 1 \\ 0 \end{array}\right\} \tag{8}$$

writing N in eigen basis and $B^T B$ in eigen basis

$$\begin{aligned} N \cdot \mathbf{B}^{\mathbf{T}}\mathbf{B}\, N &= (N_i' p_i) \cdot (\sum_{j=1}^{3} \lambda_j\, p_j \otimes p_j)(N_k' p_k) \\ &= (N_i' p_i) \cdot (\sum_{j=1}^{3} \lambda_k N_k' p_k) \\ &= \lambda_1 N_1'^2 + \lambda_2 N_2'^2 + \lambda_3 N_3'^2 \end{aligned} \tag{9}$$

also; $N_1'^2 + N_2'^2 + N_3'^2 = 1$
now since $\lambda_i > 0$

$$\begin{aligned} \left(\frac{ds}{dS}\right)^2 = \lambda_1 N_1'^2 + \lambda_2 N_2'^2 + \lambda_3 N_3'^2 &\geq \lambda_{min}(N_1'^2 + N_2'^2 + N_3'^2) \\ &= \lambda_{min} \\ &= \frac{1}{2}(2+\gamma^2-\gamma\sqrt{4+\gamma^2}) \end{aligned} \tag{10}$$

$$\begin{aligned} \left(\frac{ds}{dS}\right)^2 = \lambda_1 N_1'^2 + \lambda_2 N_2'^2 + \lambda_3 N_3'^2 &\leq \lambda_{max}(N_1'^2 + N_2'^2 + N_3'^2) \\ &= \lambda_{max} \\ &= \frac{1}{2}(2+\gamma^2+\gamma\sqrt{4+\gamma^2}) \end{aligned} \tag{11}$$

The maximum occurs
when $N_1' = N_2' = 0$ and $N_3' = 1$
implies

$$\mathbf{N} = \frac{p_3}{|p_3|} = \left\{\begin{array}{c} N_1 \\ N_2 \\ N_3 \end{array}\right\} = \frac{1}{|p_3|}\left\{\begin{array}{c} \frac{1}{2}(-\gamma-\sqrt{4+\gamma^2}) \\ 1 \\ 0 \end{array}\right\} \tag{12}$$

where $|p_3| = \frac{\sqrt{4+(\gamma+\sqrt{4+\gamma^2})^2}}{2}$
and

$$\begin{aligned}
N_1^2 - N_2^2 &= \frac{1}{|p_3|^2}\left[\frac{1}{4}(-\gamma - \sqrt{4+\gamma^2})^2 - 1\right] \\
&= \frac{1}{|p_3|^2}\left[\frac{1}{4}(2\gamma^2 + 4 - 2\gamma\sqrt{4+\gamma^2})^2 - 1\right] \\
&= \frac{1}{|p_3|^2}\left[\frac{1}{2}\gamma(\gamma + \sqrt{4+\gamma^2})\right] \\
&= -\gamma\frac{1}{|p_3|}\frac{-(\gamma + \sqrt{4+\gamma^2})}{|p_3|} \\
&= -\gamma N_1 N_2 \qquad (13)
\end{aligned}$$

2.) A body contains two families of inextensible line elements (or reinforcing fibers) aligned along the directions

$$\mathbf{M} = \cos\phi\, e_1 + \sin\phi\, e_2, \qquad \mathbf{M}^{'} = \cos\phi\, e_1 - \sin\phi\, e_2$$

($0 \le \phi \le \pi/2$) in the reference configuration, where the unit vectors $e_1,\ e_2,\ e_3$ form the fixed orthonormal basis.
If the body is subjected to the pure strain

$$\mathbf{A} = \lambda^{-1/2}\mu \mathrm{e}_1 \otimes \mathrm{e}_1 + \lambda^{-1/2}\mu^{-1}\mathrm{e}_2 \otimes \mathrm{e}_2 + \lambda \mathrm{e}_3 \otimes \mathrm{e}_3$$

show that the in-extensibility constraints lead to

$$\mu^2 = \{\lambda \pm (\lambda^2 - \sin^2 2\phi)^{1/2}\} 2\cos^2\phi.$$

Prove that
(a) if $\phi = \frac{1}{4}\pi$ then no contraction along e_3 is possible
(b) if $\lambda > \sin 2\phi$ then two deformed configurations are possible provided $\lambda \neq 1$,
(c) the maximum contraction in the e_3 direction is achieved when $\mu^2 = \tan\phi$ which correspond to the two families of fibers being mutually orthogonal in the current configuration.
Soln.

$$\mathbf{M} = \cos\phi\, e_1 + \sin\phi\, e_2 \tag{14}$$

$$\mathbf{M}^{'} = \cos\phi\, e_1 - \sin\phi\, e_2 \tag{15}$$

$$\mathbf{A} = \lambda^{-1/2}\mu \mathrm{e}_1 \otimes \mathrm{e}_1 + \lambda^{-1/2}\mu^{-1}\mathrm{e}_2 \otimes \mathrm{e}_2 + \lambda \mathrm{e}_3 \otimes \mathrm{e}_3 \tag{16}$$

Stretch along vector $\mathbf{M}$ is

$$\begin{aligned}\lambda_M &= \sqrt{\mathbf{A}\,\mathbf{M} \cdot \mathbf{A}\,\mathbf{M}} \\ &= \left[(\lambda^{-1/2}\mu\cos\phi\, e_1 + \lambda^{-1/2}\mu^{-1}\sin\phi\, e_2) \cdot (\lambda^{-1/2}\mu\cos\phi\, e_1 + \lambda^{-1/2}\mu^{-1}\sin\phi\, e_2)\right]^{1/2} \\ &= \left[\lambda^{-1}\mu^2\cos^2\phi + \frac{\lambda^{-1}}{\mu^2}\sin^2\phi\right]^{1/2} \end{aligned} \tag{17}$$

similarly,

$$\lambda_{M'} = \left[\lambda^{-1}\mu^2\cos^2\phi + \frac{\lambda^{-1}}{\mu^2}\sin^2\phi\right]^{1/2} \tag{18}$$

The condition of in-extensibility

$$\begin{aligned}\lambda_M &= 1 \\ \left[\lambda^{-1}\mu^2\cos^2\phi + \frac{\lambda^{-1}}{\mu^2}\sin^2\phi\right]^{1/2} &= 1 \\ \lambda^{-1}\mu^2\cos^2\phi + \frac{\lambda^{-1}}{\mu^2}\sin^2\phi &= 1 \\ \mu^4\cos^2\phi + \sin^2\phi - \lambda\mu^2 &= 0\end{aligned}$$

solving the quadratic equation in μ^2

$$\mu^2 = \frac{\lambda \pm \sqrt{\lambda^2 - \sin^2(2\phi)}}{2\cos^2\phi} \tag{19}$$

(a) if $\phi = \pi/4$

$$\mu^2 = \lambda + \sqrt{\lambda^2 - 1} \tag{20}$$

extension along e_3 is λ
For μ^2 to have a real solution

$$\begin{aligned}
&\lambda^2 - 1 \geq 0 \\
&\Rightarrow \lambda^2 \geq 1 \\
&\Rightarrow \lambda \geq 1 \;\; (\lambda \geq 0) \\
&\text{for contraction } \lambda < 1 \\
&\Rightarrow \text{no contraction along } e_3 \text{ possible}
\end{aligned}$$

(b) If $\lambda > \sin 2\phi$

$$\begin{aligned}
&\lambda^2 - \sin 2\phi > 0 \\
&\Rightarrow \mu^2 = \frac{\lambda \pm \sqrt{\lambda^2 - \sin^2(2\phi)}}{2\cos^2\phi}
\end{aligned}$$

has two solutions and
If $\lambda \neq 1$ then there is some deformation along e_3 direction, and two different values of μ ensures two different deformations
If $\lambda = 1$

$$\mu^2 = \frac{\lambda \pm \sqrt{\lambda^2 - \sin^2(2\phi)}}{2\cos^2\phi} = \frac{1 \pm \cos 2\phi}{2\cos^2\phi} = \begin{cases} \tan\phi \\ 1 \end{cases} \tag{21}$$

if $\mu = 1$ and $\lambda = 1$, no deformation Thus, only one type of deformation is possible.

$$\lambda - \sin^2 2\phi \geq 0, \;\; \text{for real solutions to exist} \tag{22}$$

(c) λ is lowest for highest contraction
when

$$\begin{aligned}
&\lambda^2 = \sin^2 2\phi \\
&\lambda = \pm \sin 2\phi \\
&\text{so;} \quad \mu^2 = \pm\frac{\sin 2\phi}{2\cos^2\phi} = \pm \tan\,\phi \\
&\text{since } \mu^2 > 0 \quad \mu^2 = \tan\phi
\end{aligned} \tag{23}$$

now; angle between $\mathbf{M}$ and $\mathbf{M}^{'}$ in current configuration is

$$\begin{aligned}
\mathbf{m}\cdot\mathbf{m}^{'} &= \mathbf{A}\,\mathbf{M}\cdot\mathbf{A}\,\mathbf{M}^{'} \\
&= \lambda^{-1}\mu^{2}\cos^{2}\phi - \frac{\lambda^{-1}}{\mu^{2}}\sin^{2}\phi \\
&= \lambda^{-1}\left(\frac{\sin\phi}{\cos\phi}\cos^{2}\phi - \frac{\cos\phi}{\sin\phi}\sin^{2}\phi\right) \\
&= 0
\end{aligned} \tag{24}$$

3). A body contains two families of fibers whose orientations $\mathbf{L}$ and $\mathbf{M}$ are preserved during deformation, i.e. there are positive scalars λ and μ such that

$$\mathbf{A}\,\mathbf{L} = \lambda\mathbf{L}, \qquad \mathbf{A}\,\mathbf{M} = \mu\mathbf{M}$$

In the reference configuration a line element is a parallel to $\mathbf{L}+\alpha\mathbf{M}$. Show that its length increases in the ratio

$$[\lambda^2 + 2\lambda\mu\alpha(\mathbf{L}\cdot\mathbf{M}) + \mu^2\alpha^2]^{1/2}[1 + 2\alpha(\mathbf{L}\cdot\mathbf{M}) + \alpha^2]^{1/2}$$

and that this value is stationary provided α satisfies

$$\lambda(\mathbf{L}\cdot\mathbf{M}) + (\lambda+\mu)\alpha + \mu\alpha^2(\mathbf{L}\cdot\mathbf{M})$$

$(\lambda \neq \mu)$

If the two Solutions of this equation are denoted by α_1 and α_2 show that $\mathbf{L}+\alpha_1\mathbf{M}$ and $\mathbf{L}+\alpha_2\mathbf{M}$ respectively have the orientations $\alpha_2\mathbf{L}+\mathbf{M}$ and $\alpha_1\mathbf{L}+\mathbf{M}$ in the current configuration. Deduce that the angles of rotation of line elements along $\mathbf{L}+\alpha_1\mathbf{M}$ and $\mathbf{L}+\alpha_2\mathbf{M}$ in the reference configurations are equal.

Soln.

$$\mathbf{A}\,\mathbf{L} = \lambda\mathbf{L}$$
$$\mathbf{A}\,\mathbf{M} = \mu\mathbf{M}$$

$$\begin{aligned} \mathbf{v} &= \frac{\mathbf{L}+\alpha\mathbf{M}}{[(\mathbf{L}+\alpha\mathbf{M})\cdot(\mathbf{L}+\alpha\mathbf{M})]^{1/2}} \\ &= \frac{\mathbf{L}+\alpha\mathbf{M}}{1+\alpha^2+2\alpha(\mathbf{L}\cdot\mathbf{M})} \end{aligned} \tag{25}$$

the stretch along v is

$$\begin{aligned} \lambda_v &= \sqrt{\mathbf{A}\,\mathbf{v}\cdot\mathbf{A}\,\mathbf{v}} \\ &= \frac{\sqrt{\mathbf{A}(\mathbf{L}+\alpha\mathbf{M})\cdot\mathbf{A}(\mathbf{L}+\alpha\mathbf{M})}}{[1+\alpha^2+2\alpha(\mathbf{L}\cdot\mathbf{M})]^{1/2}} \\ &= \frac{\sqrt{(\lambda\mathbf{L}+\alpha\mu\mathbf{M})\cdot(\lambda\mathbf{L}+\alpha\mu\mathbf{M})}}{[1+\alpha^2+2\alpha(\mathbf{L}\cdot\mathbf{M})]^{1/2}} \\ &= \frac{\sqrt{[\boldsymbol{\lambda^2+\mu^2+2\alpha\lambda\mu}\mathbf{L}\cdot\mathbf{M}]}}{[1+\alpha^2+2\alpha(\mathbf{L}\cdot\mathbf{M})]^{1/2}} \end{aligned} \tag{26}$$

the stationary value for λ_v implies $\frac{d\lambda_v}{d\alpha} = 0$

$$\begin{aligned} &\frac{d\lambda_v}{d\alpha} = 0 \\ &\Rightarrow \frac{2(\mathbf{L}\cdot\mathbf{M})\lambda\mu + 2\alpha\mu^2}{2\sqrt{[\boldsymbol{\lambda^2+\mu^2+2\alpha\lambda\mu}\mathbf{L}\cdot\mathbf{M}]}\sqrt{1+\alpha^2+2\alpha(\mathbf{L}\cdot\mathbf{M})}} - \frac{(2(\mathbf{L}\cdot\mathbf{M})+2\alpha)\sqrt{\boldsymbol{\lambda^2+\mu^2+2\alpha\lambda\mu}\mathbf{L}\cdot\mathbf{M}}}{2[1+\alpha^2+2\alpha(\mathbf{L}\cdot\mathbf{M})]^{3/2}} = 0 \\ &-2(\mathbf{L}\cdot\mathbf{M})\lambda^2 - 2\alpha\lambda^2 + 2(\mathbf{L}\cdot\mathbf{M})\lambda\mu - 2(\mathbf{L}\cdot\mathbf{M})\alpha^2\lambda\mu + 2\alpha\mu^2 + 2(\mathbf{L}\cdot\mathbf{M})\alpha^2\mu^2 = 0 \end{aligned} \tag{27}$$

if $\mu \neq \lambda$ the equation reduces to

$$\lambda(\mathbf{L}\cdot\mathbf{M}) + (\lambda+\mu)\alpha + \mu\alpha^2(\mathbf{L}\cdot\mathbf{M}) = 0 \tag{28}$$

the two solutions are given by α_1 and α_2
but from the quadratic equation for α

$$\alpha_1\alpha_2 = \frac{\lambda}{\mu}$$
$$\alpha_2 = \frac{\lambda}{\alpha_1\mu}$$
$$\alpha_1 = \frac{\lambda}{\alpha_2\mu}$$

$\mathbf{L} + \alpha_\mathbf{1}\mathbf{M}$ gets mapped to

$$\mathbf{A}(\mathbf{L}+\alpha_\mathbf{1}\mathbf{M}) = \lambda\mathbf{L} + \alpha_\mathbf{1}\mu\mathbf{M} = \lambda\mathbf{L} + \frac{\lambda}{\alpha_\mathbf{2}\mu}\mu\mathbf{M} = \frac{\lambda}{\alpha_\mathbf{2}}(\alpha_\mathbf{2}\mathbf{L}+\mathbf{M}) \tag{29}$$

$\mathbf{L} + \alpha_\mathbf{1}\mathbf{M}$ gets mapped along $\alpha_\mathbf{2}\mathbf{L} + \mathbf{M}$

and $\mathbf{L} + \alpha_\mathbf{2}\mathbf{M}$ gets mapped to

$$\mathbf{A}(\mathbf{L}+\alpha_\mathbf{2}\mathbf{M}) = \lambda\mathbf{L} + \alpha_\mathbf{2}\mu\mathbf{M} = \lambda\mathbf{L} + \frac{\lambda}{\alpha_\mathbf{1}\mu}\mu\mathbf{M} = \frac{\lambda}{\alpha_\mathbf{1}}(\alpha_\mathbf{1}\mathbf{L}+\mathbf{M}) \tag{30}$$

$\mathbf{L} + \alpha_\mathbf{2}\mathbf{M}$ gets mapped along $\alpha_\mathbf{1}\mathbf{L} + \mathbf{M}$