EN221 - Fall2008 - HW # 5 Solutions

Prof. Vivek Shenoy

1). For the bending of a rectangular block on page 114 of Ogden, The deformation is described by

$$r = f(X_1), \qquad \theta = g(X_2), \qquad z = \lambda X_3$$

(a) Derive the expression for the deformation gradient (second equation on page 115).
(b) Find the stretch tensors U, V and the rotation tensor R.
(c) For iso-choric deformations, derive Equation 2.2.80. Discuss why this relation is different from the result for pure bending derived in class.
Soln.
(a)

$$\bar{\mathbf{x}} = r\,e_r + z\,e_z \tag{1}$$

Deformation gradient is

$$\begin{aligned}\mathbf{A} &= \frac{\partial \bar{x}}{\partial X_i} \otimes E_i \\ &= \frac{\partial (r\,e_r + z\,e_z)}{\partial X_i} \otimes E_i \\ &= \frac{\partial r}{\partial X_i} e_r \otimes E_i + r\frac{\partial e_r}{\partial X_i} \otimes E_i + \frac{\partial z}{\partial X_i} e_z \otimes E_i \end{aligned} \tag{2}$$

now;

$$\begin{aligned}\frac{\partial r}{\partial X_i} &= \frac{df(X_1)}{dX_1} \\ \frac{\partial \theta}{\partial X_i} &= \frac{dg(X_2)}{dX_2} \\ \frac{\partial z}{\partial X_i} &= \frac{d(\lambda X_3)}{dX_3} = \lambda \end{aligned} \tag{3}$$

$$\begin{aligned} e_r &= \cos\theta\, E_1 + \sin\theta\, E_2 \\ \frac{\partial e_r}{\partial X_i} &= (-\sin\theta\, E_1 + \cos\theta\, E_2)\frac{\partial \theta}{\partial X_i} \\ &= e_\theta\, \frac{dg(X_2)}{dX_2} \end{aligned} \tag{4}$$

$\Rightarrow$

$$\mathbf{A} = \frac{df(X_1)}{dX_1} e_r \otimes E_1 + f(X_1)\frac{dg(X_2)}{dX_2} e_\theta \otimes E_2 + \lambda e_z \otimes E_3 \tag{5}$$

(b)

$$\begin{aligned}
\mathbf{U^2} &= \mathbf{A}^{\mathbf{T}}\mathbf{A} = \ f^{'2}(X_1)E_1 \otimes E_1 + f^2(X_1)g^{'2}(X_2)E_2 \otimes E_2 + \lambda^2 E_3 \otimes E_3 \\
\mathbf{U} &= f^{'}(X_1)E_1 \otimes E_1 + f(X_1)g^{'}(X_2)E_2 \otimes E_2 + \lambda E_3 \otimes E_3 \\
\mathbf{V^2} &= \mathbf{A}\,\mathbf{A}^{\mathbf{T}} = \ f^{'2}(X_1)e_r \otimes e_r + f^2(X_1)g^{'2}(X_2)e_\theta \otimes e_\theta + \lambda^2 e_z \otimes e_z \\
\mathbf{V} &= f^{'}(X_1)e_r \otimes e_r + f(X_1)g^{'}(X_2)e_\theta \otimes e_\theta + \lambda e_z \otimes e_z
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathbf{R} &= \mathbf{A}\,\mathbf{U^{-1}} \\
\mathbf{U^{-1}} &= \frac{1}{f^{'}(X_1)}E_1 \otimes E_1 + \frac{1}{f(X_1)g^{'}(X_2)}E_2 \otimes E_2 + \frac{1}{\lambda}E_3 \otimes E_3 \\
\mathbf{R} &= e_r \otimes E_1 + e_\theta \otimes E_2 + e_z \otimes E_3
\end{aligned} \tag{7}$$

(c)

$$\det \mathbf{A} = [\mathbf{A}/,E_1, \mathbf{A}/,E_2, \mathbf{A}/,E_3] \tag{8}$$

now;

$$\begin{aligned}
\mathbf{A}\,E_1 &= f^{'}(X_1)e_r \\
\mathbf{A}\,E_2 &= f(X_1)g^{'}(X_2)e_\theta \\
\mathbf{A}\,E_3 &= \lambda e_z
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\det \mathbf{A} &= [f^{'}(X_1)e_r, f(X_1)g^{'}(X_2)e_\theta, \lambda e_z] \\
&= f^{'}(X_1)f(X_1)g^{'}(X_2)\lambda[e_r, e_\theta, e_z] \\
&= f^{'}fg^{'}\lambda
\end{aligned} \tag{9}$$

for iso-choric deformation

$$\begin{aligned}
& f^{'}fg^{'}\lambda = 1 \\
& f^{'}f = \frac{1}{g^{'}\lambda} \\
& f\frac{df(X_1)}{dX_1} = \frac{1}{\lambda\frac{dg(X_2)}{dX_2}} = \text{constant} = \frac{1}{\alpha}
\end{aligned} \tag{10}$$

$\Rightarrow$

$$
\begin{aligned}
\frac{f^2(X_1)}{2} &= \frac{X_1}{\alpha} + \frac{\beta}{2} \\
r^2 = f^(X_1) &= \frac{2X_1}{\alpha} + \beta
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
\frac{\alpha}{\lambda} &= \frac{dg(X_2)}{dX_2} \\
\theta = g(x_2) &= \frac{\alpha X_2}{\lambda}
\end{aligned} \tag{12}
$$

This relation is different from that obtained in the class for the following reasons
(1) det $\mathbf{A} = 1$ (in-compressibility) in this case as opposed to the example in the class
(2) In the class example stretch along X_1 is 1, while it in this case it's not equal to 1
(3) In class, stretch λ along the X_2 at $X_1 = 0$ is 1. In this case its not true

2.) An initially planar sheet of deform-able material is to be wrapped uniformly around a non-deform-able cylinder as shown in the figure below (like a rubber insulating layer wrapped around a metal wire). Assume that the deformation is plane strain, so that strains involving the 3-component vanish. The initial length of the sheet is equal the circumference of the cylinder, πD, where D is the diameter of the cylinder.
(a) With the choice of coordinate axes and origin shown in the sketch, view the de- formation as a compound (2-stage) deformation, $X \to q \to x$. First, impose a non-uniform extension of the sheet in the 2-direction given by

$$q = X_1 e_1 + h(X_2) e_2 + X_3 e_3$$

where $h(X_2)$ is a function to be determined, subject to the condition $h(0) = 0$.This deformation is followed by pure bending, so that the sheet wraps around the cylinder as shown in part (b) of the figure. By writing the final deformed coordinates x in terms of X, calculate the deformation gradient F and the stretch tensor U. Note that U should depend on $h(X_2)$ and $h^{'}(X2)$, where prime denotes differentiation with respect to X_2.
(b) Assume that there is no volume change at any point in the body. From the condition that $J = \det(F) = 1$, determine the ordinary differential equation that the function $h(X_2)$ must satisfy, and solve that equation.
Soln.
We write the deformation gradient in cylindrical coordinates

$$R = \frac{D}{2} + h(X_2) \tag{13}$$

Now, since it is pure bending planes perpendicular to X_1direction remain perpendicular on deformations. Means θ is independent of X_2 and thus

$$\theta = \frac{X_1}{D/2} \tag{14}$$

$$\bar{x} = R\, e_r + z\, e_z \tag{15}$$

The deformation gradient is

$$\begin{aligned}
\mathbf{F} &= \nabla \otimes \bar{\mathrm{x}} \\
&= \frac{\partial \bar{x}}{\partial X_1} \otimes E_1 + \frac{\partial \bar{x}}{\partial X_2} \otimes E_2 + \frac{\partial \bar{x}}{\partial X_3} \otimes E_3 \\
&= R\frac{\partial e_R}{\partial X_1} \otimes E_1 + R\frac{\partial e_R}{\partial X_2} \otimes E_2 + \frac{\partial R}{\partial X_2} e_R \otimes E_2 + \frac{\partial z}{\partial X_3} e_z \otimes E_3 \\
&= R\frac{d\theta}{dX_1} e_\theta \otimes E_1 + R\frac{d\theta}{dX_2} \otimes E_2 + \frac{\partial R}{\partial X_2} e_R \otimes E_2 + \frac{\partial z}{\partial X_3} e_z \otimes E_3 \\
&= (\frac{D}{2} + h(X_2))\frac{2}{D} e_\theta \otimes E_1 + h(X_2) e_R \otimes E_2 + e_z \otimes E_3
\end{aligned}$$

$$\mathbf{U^2} = \mathbf{F^{^T} F}$$

$$\mathbf{U} = \left(1 + \frac{2h(X_2)}{D}\right) E_1 \otimes E_1 + \frac{dh(X_2)}{dX_2} E_2 \otimes E_2 + E_3 \otimes E_3 \tag{16}$$

(b)

$$\begin{aligned} &\det \mathbf{U} = \det \mathbf{F} = J = 1 \\ &= \left(1 + \frac{2h(X_2)}{D}\right) h^{'}(X_2) = 1 \\ &\frac{D}{4}\left(1 + \frac{2h(X_2)}{D}\right)^2 = X_2 + C \end{aligned} \tag{17}$$

we have $h(0) = 0$

$$\Rightarrow \frac{D}{4} = C$$

$$\Rightarrow \frac{D}{4}\left(1 + \frac{2h(X_2)}{D}\right)^2 = X_2 + \frac{D}{4}$$

$$\Rightarrow h(x_2) = \frac{D}{2}\left[-1 + \sqrt{1 + \frac{4X_2}{D}}\right]$$

3). Consider the combined axial and azimuthal shear deformation of a circular tube defined by

$$r = R, \qquad \theta = \Theta + \phi(R), \qquad z = Z + w(R)$$

where upper(lower) case symbols indicate undeformed(deformed) coordinates.
(a) Calculate the deformation gradient in cylindrical coordinates.
(b) Obtain the principal stretches and verify that the deformation is iso-choric ($J = 1$).
Soln.
Deformation is given by

$$r = R, \qquad \theta = \Theta + \phi(R), \qquad z = Z + w(R)$$

$$\bar{x} = re_r + ze_z$$

Consider deformation gradient in cylindircal coordinates

$$\begin{aligned}
\mathbf{F} &= \nabla \otimes \bar{x} \\
&= \frac{\partial re_r + ze_z}{\partial R} \otimes E_R + \frac{\partial re_r + ze_z}{\partial \Theta} \otimes E_\Theta + \frac{\partial re_r + ze_z}{\partial Z} \otimes E_Z \\
&= e_r \otimes E_R + R\frac{\partial e_r}{\partial R} \otimes E_R + \frac{dw(R)}{dR} e_z \otimes E_R + \frac{r}{R}\frac{\partial e_r}{\partial \Theta} \otimes E_\Theta + e_z \otimes E_Z
\end{aligned}$$

$$\frac{\partial e_r}{\partial R} = e_\theta \frac{d\Theta}{dR} = e_\theta \frac{d\phi(R)}{dR}$$

$$\frac{\partial e_r}{\partial \Theta} = e_\theta \frac{d\theta}{d\Theta} = e_\theta$$

Thus,

$$\mathbf{F} = e_r \otimes E_R + R\frac{d\phi(R)}{dR} e_\theta \otimes E_R + \frac{dw(R)}{dR} e_z \otimes E_R + e_\theta \otimes E_\Theta + e_z \otimes E_Z \tag{18}$$

(b)

$$\begin{aligned}
\mathbf{U^2} &= \mathbf{F^T F} \\
&= \begin{bmatrix} 1 + w'^2 + (R\phi')^2 & R\phi' & w' \\ R\phi' & 1 & 0 \\ w' & 0 & 1 \end{bmatrix}
\end{aligned} \tag{19}$$

$$\begin{aligned}
\det \mathbf{U^2} &= (\det \mathbf{F})^2 = J^2 \\
&= (1 + w'^2 + (R\phi')^2) - R\phi' R\phi' + w'(-w') \\
&= 1
\end{aligned} \tag{20}$$

Hence $J = 1$ imples the deformation is iso-choric
To find the principal stretches we find eigenvalues of $\mathbf{U^2}$. The principal stretches are square roots of those values

$$(1 + w^{'2} + (R\phi^{'})^2 - \lambda)(1 - \lambda)^2 - R\phi^{'}(R\phi^{'}(1 - \lambda)) + w^{'}(-w^{'}(1 - \lambda)) = 0$$
$$\left[(1 + w^{'2} + (R\phi^{'})^2 - \lambda)(1 - \lambda) - (R\phi^{'})^2 - w^{'2}\right](1 - \lambda) = 0$$
$$\left[\lambda^2 - \lambda(2 + w^{'2} + (R\phi^{'})^2) + 1\right](1 - \lambda) = 0 \tag{21}$$

On solving above Eqn for λ

$$\lambda_1 = \quad 1 \tag{22}$$
$$\lambda_{2,3} = \quad \frac{1}{2}\left[(2 + w^{'2} + (R\phi^{'})^2) \pm \sqrt{(2 + w'^2 + (R\phi')^2) - 4}\right] \tag{23}$$

So the principal stretches are $\sqrt{\lambda}_1$,$\sqrt{\lambda}_2$, and $\sqrt{\lambda}_3$

4). Show that the gradient of the spherically symmetric deformation $\mathbf{x} = f(R)\mathbf{X}$, where $R = |\mathbf{X}|$, is

$$f(R)\mathbf{I} + \frac{1}{R}f^{'}(R)\mathbf{X}\otimes\mathbf{X}$$

and find $f(R)$ if the deformation iso-choric.
A spherical shell is defined by

$$A \leq R \leq B, \qquad 0 \leq \Theta \leq \pi, \qquad 0 \leq \Phi \leq 2\pi$$

in some reference configuration where (R, Θ, Φ) are spherical polar coordinates. If the material of the shell is incompressible and the shell deformed so that spherical symmetry is maintained, show that

$$r^3 = R^3 + a^3 - A^3, \qquad \theta = \Theta, \qquad \phi = \Phi$$

where the current configuration of the shell is defined in terms of spherical polar coordinates (r, θ, ϕ) such that $a \leq r \leq b$
Deduce that the principal stretches are λ^{-2}, λ,and λ, where $\lambda = r/R$ and show that if $\lambda_a = a/A$ $\lambda_b = b/B$ then

$$\left(\frac{B}{A}\right)^3(\lambda_b^3 - 1) = \lambda_a^3 - 1$$

and hence either $\lambda_a \leq \lambda_b \leq 1$ or $\lambda_a \geq \lambda_b \geq 1$
Soln.

$$\bar{X} = R\,E_R$$
$$\bar{x} = f(R)\bar{X} = Rf(R)\,E_R$$

so, Deformation gradient in the spherical coordinates

$$\begin{aligned}
\mathbf{F} &= \nabla\otimes\bar{x} \\
&= \nabla\otimes(Rf(R)\,E_R) \\
&= \frac{\partial(Rf(R)\,E_R)}{\partial R}\otimes E_R + \frac{1}{R}\frac{\partial(Rf(R)\,E_R)}{\partial\phi}\otimes E_\Phi + \frac{1}{R\sin\phi}\frac{\partial(Rf(R)\,E_R)}{\partial\Theta}\otimes E_\Theta \\
&= f(R)E_R\otimes E_R + Rf^{'}(R)E_R\otimes E_R + f(R)\frac{\partial E_R}{\partial\phi}\otimes E_\phi + \frac{f(R)}{\sin\phi}\frac{\partial E_R}{\partial\Theta}\otimes E_\Theta
\end{aligned}$$

but,

$$\begin{aligned}
\frac{\partial E_R}{\partial\phi} &= E_\phi \\
\frac{\partial E_R}{\partial\Theta} &= \sin\phi\,E_\Theta
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\mathbf{F} &= R\,f^{'}(R)E_R \otimes E_R + f(R)\left[E_R \otimes E_R + E_\Theta \otimes E_\Theta + E_\phi \otimes E_\phi\right] \\
&= R\,f^{'}(R)E_R \otimes E_R + f(R)\mathbf{I} \\
&= \frac{R}{R^2} f^{'}(R) R\,E_R \otimes R\,E_R + f(R) \\
&= \frac{1}{R} f^{'}(R)\bar{X} \otimes \bar{X} + f(R) \qquad (24)
\end{aligned}$$

The deformation is iso-choric

$$\begin{aligned}
\det\mathbf{F} = J = 1 &= f^2(R)\left[f(R) + R\,f^{'}(R)\right] \\
\Rightarrow f(R) + R\,f^{'}(R) &= \frac{1}{f^2(R)} \\
\Rightarrow R\,f^{'}(R) &= \frac{1 - f^3(R)}{f^2(R)} \\
\Rightarrow \frac{dR}{R} &= \frac{1}{3}\frac{3f^2(R)}{[1 - f^3(R)]} \\
\Rightarrow \log\left[1 - f^3(R)\right] = -3\log R + \log\alpha & \\
\Rightarrow [1 - f^3(R)] = \frac{\alpha}{R^3} & \\
\Rightarrow f(R) = \left(1 - \frac{\alpha}{R^3}\right)^{1/3} & \qquad (25)
\end{aligned}$$

Now,
$A \rightarrow a$
$B \rightarrow b$
$\Rightarrow$

$$\bar{x} = r\,e_R = R\,f(R)\,E_R \qquad (26)$$

hence,

$$\begin{aligned}
\bar{x}(A) &= a = A\,f(A) \\
a &= A(1 - \alpha A^{-3})^{1/3} \\
\Rightarrow a^3 &= A^3 - \alpha \\
\Rightarrow \alpha &= A^3 - a^3 \qquad (27)
\end{aligned}$$

hence,

$$\begin{aligned}
r^3 &= R^3 f^3(R) \\
&= R^3(1 - \alpha R^{-3}) \\
&= R^3(1 - (A^3 - a^3)R^{-3}) \\
&= R^3 - A^3 + a^3 \qquad (28)
\end{aligned}$$

Since spherical symmetry is maintained $\theta = \Theta$ and $\phi = \Phi$
Since $\mathbf{F}$ is diagonal
the principal stretches are clearly
$f(R) + f^{'}(R)$, $f(R)$, and $f(R)$
But, $f(R) = \frac{r}{R} = \lambda$ and
Since, $J = 1$ (by assumption)

$$J = \lambda_1 \lambda_2 \lambda_3 = \lambda^2 \lambda_3 = 1$$
$$\Rightarrow \lambda_3 = \lambda^{-2}$$

if $\lambda_a = \frac{a}{A}$ and $\lambda_b = \frac{b}{B}$
we have

$$1 - f^3 = \frac{1}{\alpha R^3}$$

$\Rightarrow$

$$1 - \left(\frac{a}{A}\right)^3 = \frac{1}{\alpha A^3} \tag{29}$$
$$1 - \left(\frac{b}{B}\right)^3 = \frac{1}{\alpha B^3} \tag{30}$$

on eliminating α from above two equations
$\Rightarrow$

$$\left(\frac{B}{A}\right)^3 (\lambda_b^3 - 1) = (\lambda_a^3 - 1) \tag{31}$$

Since $B > A \Rightarrow \frac{B}{A} > 1$
if $\lambda_b^3 - 1 \geq 0$

$$\begin{aligned} \left(\frac{B}{A}\right)^3 (\lambda_b^3 - 1) &\geq (\lambda_b^3 - 1) \\ (\lambda_a^3 - 1) &\geq (\lambda_b^3 - 1) \\ \Rightarrow \lambda_a &\geq \lambda_b \geq 1 \end{aligned} \tag{32}$$

Similarly,
if $\lambda_b^3 - 1 \leq 0$

$$\begin{aligned} \left(\frac{B}{A}\right)^3 (\lambda_b^3 - 1) &\leq (\lambda_b^3 - 1) \\ (\lambda_a^3 - 1) &\leq (\lambda_b^3 - 1) \\ \Rightarrow \lambda_a &\leq \lambda_b \leq 1 \end{aligned} \tag{33}$$