EN221 - Fall2008 - HW # 6 Solutions

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1). Consider the motion of a deformable body described the velocity components (in the spatial description)

$$v1 = x2, v2 = x1, v3 = 0$$

Assume that the reference configuration of the body is chosen to be the configuration at $t = 0$, so that $X = x$ at $t = 0$
(a) Derive the particle path i.e $x(X, t)$
(b) Compute the velocity components in terms of the material coordinates X, and obtain the associated acceleration in both the material and the spatial descriptions.
(c) Since the motion is steady, the pathlines and streamlines of (4) should coincide. Derive the equations for the pathlines and streamlines separately and make sure that they are the same.
Soln.
Given $v_1 = x_2$; $v_2 = x_1$; $v_3 = 0$

$$\Rightarrow \qquad \frac{\partial x_1(X,t)}{\partial t} = x_2 \tag{1}$$

$$\Rightarrow \qquad \frac{\partial^2 x_1}{\partial t^2} = \frac{\partial x_2}{\partial t} = v_2 = x_1 \tag{2}$$

on combining above two equations

$$\frac{d^2 x_1}{dt^2} = x_1 \tag{3}$$

The general solution of the above ODE is give by

$$x_1 = A_1(X)\exp(t) + A_2(X)\exp(-t) \tag{4}$$

Boundary condition are

$$@t = 0, \;\; x_1 = X_1 \tag{5}$$

$$@t = 0, \;\; v_1 = x_2 = X_2 \tag{6}$$

On substituting the first B.C.

$$X_1 = A_1(X) + A_2(X) \tag{7}$$

On substituting the second B.C.

$$X_2 = A_1(X) - A_2(X) \tag{8}$$

on solving above two equations

$$A_1(X) = \frac{X_1 + X_2}{2} \tag{9}$$

$$A_2(X) = \frac{X_1 - X_2}{2} \tag{10}$$

hence,

$$\begin{aligned}
x_1(t) &= \frac{X_1 + X_2}{2}\exp(t) + \frac{X_1 - X_2}{2}\exp(-t) && (11)\\
x_2(t) &= \frac{X_1 + X_2}{2}\exp(t) - \frac{X_1 - X_2}{2}\exp(-t) && (12)\\
x_3(t) &= X_3 && (13)
\end{aligned}$$

The velocity in terms of material co-ordinates is

$$\begin{aligned}
v_1 &= x_2(t) = \frac{X_1 + X_2}{2}\exp(t) - \frac{X_1 - X_2}{2}\exp(-t) && (14)\\
v_2 &= x_1(t) = \frac{X_1 + X_2}{2}\exp(t) + \frac{X_1 - X_2}{2}\exp(-t) && (15)\\
v_3 &= 0 && (16)
\end{aligned}$$

Acceleration in material coordinates

$$\begin{aligned}
a_1(t) &= \frac{X_1 + X_2}{2}\exp(t) + \frac{X_1 - X_2}{2}\exp(-t) && (17)\\
a_2(t) &= \frac{X_1 + X_2}{2}\exp(t) - \frac{X_1 - X_2}{2}\exp(-t) && (18)\\
a_3(t) &= 0 && (19)
\end{aligned}$$

In spatial cordinates

$$\begin{aligned}
a_1(t) &= x_1 && (20)\\
a_2(t) &= x_2 && (21)\\
a_3(t) &= 0 && (22)
\end{aligned}$$

(c)
Motion is clearly steady
Streamlines are the curves with tangent along the direction of velocity

$$\text{i.e. } \frac{dF}{dr} = \frac{\bar{v}}{|v|}$$

$$\Rightarrow \frac{dx_1}{ds} = \frac{v_1}{|v|} \tag{23}$$

$$\frac{dx_2}{ds} = \frac{v_2}{|v|} \tag{24}$$

$$\Rightarrow \frac{dx_1}{v_1} = \frac{dx_2}{v_2} \tag{25}$$

$$\Rightarrow x_1\, dx_1 = x_2\, dx_2 \tag{26}$$

$$\Rightarrow x_1^2 - x_2^2 = \alpha = \text{constant} \tag{27}$$

Similarly,

$$x_3 = \beta = \text{constant} \tag{28}$$

where α gives family of curves
Path line

$$x_1(t) = \frac{X_1 + X_2}{2} \exp(t) + \frac{X_1 - X_2}{2} \exp(-t) \tag{29}$$

$$x_2(t) = \frac{X_1 + X_2}{2} \exp(t) - \frac{X_1 - X_2}{2} \exp(-t) \tag{30}$$

$\Rightarrow$

$$x_1^2 - x_2^2 = X_1^2 - X_2^2 \tag{31}$$

$$x_3 = X_3 \tag{32}$$

so now if we put $\alpha = X_1^2 - X_2^2$ and $\beta = X_3$
(for every α we can find (X_1, X_2) and vice-versa)
we obtain the same curve for path line and streamline.

2.) The velocity field in a motion of a body is given, in the spatial description and relative to a rectangular coordinate system, by

$$v_1 = -n \sin\, nt\ (2 + \cos\, nt)^{-1} x_1$$

$$v_2 = n \cos\, nt\ (2 + \sin\, nt)^{-1} x_2$$

where n is a positive constant. Choosing as reference configuration the placement of the body at $t = 0$ and adopting a common rectangular Cartesian coordinate system, obtain expressions for the x_i in terms of referential coordinates X_α and t. Determine the particle paths and the streamlines of the motion and discuss the relationship between them. Find the volume v at the time t of a material region which has volume V in the reference configuration, and show that the greatest and least values of v are $(\frac{3}{4} \pm \frac{1}{3}\sqrt{2})V$
Soln.

$$\begin{aligned} v_1 &= \frac{dx_1}{dt} = \frac{-n \sin nt}{2 + \cos nt} x_1 \\ \Rightarrow \ln x_1 &= \ln\left[(2 + \cos nt)\alpha\right] \\ \Rightarrow x_1 &= \alpha[2 + \cos nt] \end{aligned} \tag{33}$$

at $t = 0$, $x_1 = X_1$

$$\Rightarrow x_1 = \frac{X_1}{3}[2 + \cos nt] \tag{34}$$

Similarly,

$$\Rightarrow x_2 = \frac{X_2}{2}[2 + \sin nt] \tag{35}$$

and

$$x_3 = X_3 \tag{36}$$

Eliminating t between Eqn(33) and (34) we obtain path line

$$\left(\frac{3x_1}{X_1} - 2\right)^2 + \left(\frac{2x_2}{X_2} - 2\right)^2 = 1 \tag{37}$$

$$x_3 = X_3 \tag{38}$$

The motion is unsteady, because the velocity depends also on time. To obtain streamline, we follow same procedure as in the previous problem

$$\begin{aligned} \frac{dx_1}{v_1} &= \frac{dx_2}{v_2} \\ \Rightarrow \frac{2 + \cos nt}{-n \sin nt}\frac{dx_1}{x_1} &= \frac{2 + \sin nt}{n \cos nt}\frac{dx_2}{x_2} \\ \Rightarrow \ln x_1 &= f(t) \ln x_2 + \ln \beta \\ \text{where } \ f(t) &= \frac{-\sin nt}{\cos nt}\frac{2 + \sin nt}{2 + \cos nt} \end{aligned}$$

$$x_1 = \beta x_2^{f(t)} \tag{39}$$

$$x_3 = \alpha \tag{40}$$

is the family of streamlines as a function of time.
The streamlines depend on time. Even at any given instant of time the streamlines obey a power-law, while the path lines have a periodic motion along a ellipse.
The deformation gradient with basis as E_1, E_2, and E_3

$$\mathbf{F}(t) = \begin{bmatrix} \frac{(2+\cos nt)}{3} & 0 & 0 \\ 0 & \frac{2+\sin nt}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{41}$$

$$\begin{aligned} \frac{v}{V} = J(t) &= \det \mathbf{F} = \frac{1}{6}(2+\cos nt)(2+\sin nt) \\ &= \frac{1}{6}[4 + 2(\cos nt + \sin nt) + \frac{1}{2}\sin 2nt] \end{aligned}$$

$\Rightarrow$ the max./min. values of the above expression is obtained as follows

$$\begin{aligned} \frac{dJ}{d(nt)} &= 0 \\ \frac{1}{6}[2(-\sin nt + \cos nt) + \cos 2nt] &= 0 \\ \Rightarrow [2(\cos nt - \sin nt) + \cos^2 nt - \sin^2 nt] &= 0 \\ \Rightarrow (\cos nt - \sin nt)[2 + \cos nt + \sin nt] &= 0 \end{aligned} \tag{42}$$

Since $[2 + \cos nt + \sin nt] \neq 0$ always
hence $\cos nt = \sin nt$

$$\Rightarrow \quad nt = \left\{ \frac{\pi}{4}, \frac{5\pi}{4}, \cdots \right\}$$

greatest value $@nt = \frac{\pi}{4}$

$$\Rightarrow J = (\frac{3}{4} + \frac{1}{3}\sqrt{2})$$

least value $@nt = \frac{5\pi}{4}$

$$\Rightarrow J = (\frac{3}{4} - \frac{1}{3}\sqrt{2})$$

3).Obtain the formulae

$$\dot{(da)} = \{\mathrm{tr}\,\mathbf{L} - \mathbf{n}\cdot(\mathbf{L}\mathbf{n})\}da$$

$$\dot{\mathbf{n}} = \{\mathbf{n}\cdot(\mathbf{L}\mathbf{n})\}\mathbf{n} - \mathbf{L}^{\mathrm{T}}\mathbf{n}$$

relating to a material surface element of area da with unit normal $\mathbf{n}$, $\mathbf{L}$ being the velocity gradient at the current location of the element
Soln.
To show the first result, we use Eqn 39 Pg 67 of Chadwick
Substitute $\mathbf{u}$, with $\mathbf{n}$

$$\begin{aligned} \frac{d}{dt}\int_{S_t}\mathbf{n}\cdot\mathbf{n}da &= \int_{S_t}(\dot{\mathbf{n}} + \mathbf{n}\mathrm{tr}\,\mathbf{L} - \mathbf{L}\mathbf{n})\cdot\mathbf{n}\,da \\ \frac{d}{dt}\int_{S_t}1da = \int_{S_t}\frac{d(da)}{dt} &= \int_{S_t}(\dot{\mathbf{n}} + \mathbf{n}\mathrm{tr}\,\mathbf{L} - \mathbf{L}\mathbf{n})\cdot\mathbf{n}\,da \end{aligned} \tag{43}$$

since S_t is arbitrary, also $\mathbf{n}\cdot\mathbf{n} = 1$, hence

$$\dot{(\mathbf{n}\cdot\mathbf{n})} = 2\,\dot{\mathbf{n}}\cdot\mathbf{n} = \mathbf{0}$$

$$\begin{aligned} \dot{(da)} &= \{\dot{\mathbf{n}}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{n}\,\mathrm{tr}\,\mathbf{L} - \mathbf{L}\mathbf{n}\cdot n\}da \\ &= \{\mathrm{tr}\,\mathbf{L} - (\mathbf{L}\mathbf{n})\cdot\mathbf{n}\}da \end{aligned} \tag{44}$$

Now, to prove the second identity, we make use of the following
Use Eqn 36, with $\phi = 1$ from Chadwick

$$\begin{aligned} \frac{d}{dt}\int_{S_t}\mathbf{n}\,da &= \int_{S_t}\Big[(\mathrm{tr}\,\mathbf{L})\mathbf{n} - \mathbf{L}^{\mathrm{T}}\mathbf{n}\Big]\,da \\ \int_{S_t}\frac{d(\mathbf{n}\,da)}{dt} = \int_{S_t}(\dot{\mathbf{n}}\,da + (\dot{da})\,\mathbf{n}) &= \int_{S_t}\Big[(\mathrm{tr}\,\mathbf{L})\mathbf{n} - \mathbf{L}^{\mathrm{T}}\mathbf{n}\Big]\,da \end{aligned}$$

Since S_t is arbitrary

$$(\dot{\mathbf{n}}\,da + (\dot{da})\,\mathbf{n}) = \Big[(\mathrm{tr}\,\mathbf{L})\mathbf{n} - \mathbf{L}^{\mathrm{T}}\mathbf{n}\Big]\,da \tag{45}$$

now using Eqn(44) in the above Eqn and simplifying

$$\begin{aligned} \dot{\mathbf{n}}\,da + [\mathrm{tr}\,\mathbf{L} - (\mathbf{L}\mathbf{n})\cdot\mathbf{n}]\,da\,\mathbf{n} &= \Big[(\mathrm{tr}\,\mathbf{L})\mathbf{n} - \mathbf{L}^{\mathrm{T}}\mathbf{n}\Big]\,da \\ \Rightarrow \dot{\mathbf{n}} = \{\mathbf{n}\cdot(\mathbf{L}\mathbf{n})\}\mathbf{n} - \mathbf{L}^{\mathrm{T}}\mathbf{n} & \end{aligned} \tag{46}$$