EN221 - Fall2008 - HW # 7 Solutions

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1.) Show that the formulae

$$\frac{d}{dt}\int_{R_t}\phi\,dv = \int_{R_t}(\dot{\phi}+\phi \mathrm{tr}\,\mathbf{L})dv \tag{1}$$

$$\frac{d}{dt}\int_{R_t}\mathbf{u}\,dv = \int_{R_t}(\dot{\mathbf{u}}+\mathbf{u}\mathrm{tr}\,\mathbf{L})dv \tag{2}$$

can be put into the alternative forms

$$\frac{d}{dt}\int_{R_t}\phi\,dv = \int_{R_t}\frac{\partial\phi}{\partial t}dv + \int_{\partial R_t}\phi\mathbf{v}\cdot\mathbf{n}da \tag{3}$$

$$\frac{d}{dt}\int_{R_t}\mathbf{u}\,dv = \int_{R_t}\frac{\partial\mathbf{u}}{\partial t}dv + \int_{\partial R_t}\mathbf{u}(\mathbf{v}\cdot\mathbf{n})da \tag{4}$$

Soln.
(a)
Using $dv = JdV$, and changing the integral to the reference frame.

$$\begin{aligned}
\frac{d}{dt}\int_{R_t}\phi\,dv &= \int_{R_r}\frac{d}{dt}(\phi JdV) \\
&= \int_{R_r}(\dot{\phi}J+\phi\dot{J})dV \\
&= \int_{R_r}(\dot{\phi}J+\phi J\mathrm{tr}\,\mathbf{L})dV \\
&= \int_{R_r}(\dot{\phi}+\phi\,\mathrm{tr}\,\mathbf{L})JdV \\
&= \int_{R_t}(\dot{\phi}+\phi\,\mathrm{div}\,\mathbf{v})dv
\end{aligned} \tag{5}$$

but,

$$(\mathrm{grad}\,\phi)\cdot\mathbf{v}+\phi\,\mathrm{div}\,\mathbf{v} = \mathrm{div}\,(\phi\,\mathbf{v})$$

$$\dot{\phi} = \frac{\partial\,\phi}{\partial\,t}+(\mathrm{grad}\,\phi)\cdot\mathbf{v}$$

$$\frac{d}{dt}\int_{R_t}\phi\,dv = \int_{R_t}(\dot{\phi}+\phi\,\mathrm{div}\,\mathbf{v})dv$$

$$
\begin{aligned}
&= \int_{R_t} \left(\frac{\partial \phi}{\partial t} + (\operatorname{grad} \phi) \cdot \mathbf{v} + \phi \operatorname{div} \mathbf{v} \right) dv \\
&= \int_{R_t} \left(\frac{\partial \phi}{\partial t} + \operatorname{div} (\phi \mathbf{v}) \right) dv \qquad (6)
\end{aligned}
$$

Using divergence theorem

$$
\frac{d}{dt} \int_{R_t} \phi \, dv = \int_{R_t} \frac{\partial \phi}{\partial t} + \int_{\partial R_t} \phi \, \mathbf{v} \cdot \mathbf{n} \, da \qquad (7)
$$

(b) Using $dv = JdV$, and changing the integral to the reference frame.

$$
\begin{aligned}
\frac{d}{dt} \int_{R_t} \mathbf{u} dv &= \int_{R_r} \frac{d(\mathbf{u}J)}{dt} dV \\
&= \int_{R_r} (\dot{\mathbf{u}} J + \mathbf{u} \dot{J}) dV \\
&= \int_{R_r} (\dot{\mathbf{u}} J + \mathbf{u} J \operatorname{tr} \mathbf{L}) dV \\
&= \int_{R_r} (\dot{\mathbf{u}} + \mathbf{u} \operatorname{tr} \mathbf{L}) J dV \\
&= \int_{R_t} (\dot{\mathbf{u}} + \mathbf{u} \operatorname{div} \mathbf{v}) dv \qquad (8)
\end{aligned}
$$

but,

$$
(\operatorname{grad} \mathbf{u}) \cdot \mathbf{v} + \mathbf{u} \operatorname{div} \mathbf{v} = \operatorname{div} (\mathbf{v} \otimes \mathbf{u})
$$

$$
\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} + (\operatorname{grad} \mathbf{u}) \cdot \mathbf{v}
$$

$$
\begin{aligned}
\frac{d}{dt} \int_{R_t} \mathbf{u} dv &= \int_{R_t} (\dot{\mathbf{u}} + \mathbf{u} \operatorname{div} \mathbf{v}) dv \\
&= \int_{R_t} \left(\frac{\partial \mathbf{u}}{\partial t} + (\operatorname{grad} \mathbf{u}) \cdot \mathbf{v} + \mathbf{u} \operatorname{div} \mathbf{v} \right) dv \\
&= \int_{R_t} \left(\frac{\partial \mathbf{u}}{\partial t} + \operatorname{div} (\mathbf{v} \otimes \mathbf{u}) \right) dv \qquad (9)
\end{aligned}
$$

Using divergence theorem

$$
\frac{d}{dt} \int_{R_t} \mathbf{u} \, dv = \int_{R_t} \frac{\partial \mathbf{u}}{\partial t} + \int_{\partial R_t} \mathbf{u} \, (\mathbf{v} \cdot \mathbf{n}) \, da \qquad (10)
$$

2.) Consider the cutting configuration shown in the figure below. The workpiece is assumed to be an incompressible material of mass density ρ. The opening angle of the stationary cutting tool is α. The work piece is pushed towards the tool by a force of magnitude F, at a constant speed V_0. The thickness (length in the x_3 direction) of the workpiece is b (not shown in the figure). The contact between the chip and tool obeys Coulomb friction with coefficient of friction μ, so that $R = \mu N$, where R and N are the frictional and normal-reaction forces, respectively, as shown in the figure. Note that the chip emerges at a speed V and makes an angle β to the horizontal.
(a) Using conservation of mass, relate the chip speed V and the cutting speed V_0.
(b) Using linear momentum balance, compute the cutting force F and the tool reactions N and R in terms of ρ, b, h, V_0, α, β and μ.
Soln.
from the figure $\angle DAB + \alpha + \beta = \pi$
$\angle DAB = \pi - (\alpha + \beta)$
also it can be seen that

$$\sin \angle DAB = \sin[\pi - (\alpha+\beta)] = \frac{c}{BA} \quad (11)$$
$$\sin\beta = \frac{h}{BA} \quad (12)$$

from the Eqns(11 and 12)

$$\frac{\sin(\alpha+\beta)}{\sin\beta} = \frac{c}{h}$$
$$c = h\frac{\sin(\alpha+\beta)}{\sin\beta} \quad (13)$$

let $d(BE) = x$ and $d(BF) = l$ so the total mass of the workpiece is (which is incompressible)
Note: ρ is constant

$$M = \rho x h b + \rho l c b + \text{mass of small region in between(constant)}$$
$$\frac{dM}{dt} = 0$$
$$\Rightarrow 0 = \rho \dot{x} h b + \rho \dot{l} c h$$
$$\Rightarrow v = \dot{l} = \frac{-\dot{x}h}{c} = \frac{V_o h}{c} \quad (\dot{x} = -V_o) \quad (14)$$

The momentum is

$$P = \rho\, v\, l\, c\, b\, \cos\alpha\, e_1 + \rho\, v\, l\, c\, b\, \sin\alpha\, e_2 + \rho\, V_o\, x\, c\, b\, e_1 + \text{momentum of smallregion(constant)}$$

$$\frac{dP}{dt} = \text{Force} = [F - N\mu\cos\alpha - N\sin\alpha]e_1 + [N\cos\alpha + N\mu\sin\alpha]e_2 \quad (15)$$
$$\frac{dP}{dt} = (\rho\, V^2 c\, b\cos\alpha + \rho\, V_0^2 h\, b)e_1 + (\rho\, V^2 c\, b \sin\alpha)e_2 \quad (16)$$

on comparing the Eqns(15 and 16) we obtian

$$N = \frac{\rho V^2 c\, b\, \sin\alpha}{\mu \sin\alpha - \cos\alpha} \tag{17}$$

$$R = N\mu \tag{18}$$

$$F = \rho\, b \left[\frac{V^2 c \sin\alpha(\mu\cos\alpha + \sin\alpha)}{\mu\sin\alpha - \cos\alpha} + V_0^2 - V^2 c \right] \tag{19}$$

3).If $\mathbf{t}$ denotes the traction per unit area on a surface whose normal is in direction $\mathbf{n}$, show that the square of the magnitude of the shear stress on that surface is

$$\mathbf{t} \cdot \mathbf{t} - (\mathbf{n}.\mathbf{t})^{\mathbf{2}}$$

If n_i $(i = 1, 2, 3)$ are the components of $\mathbf{n}$ relative to the principal axes of the Cauchy stress tensor, show that the above expression may be written

$$(t_2 - t_3)^2 n_2^2 n_3^2 + (t_3 - t_1)^2 n_3^2 n_1^2 + (t_1 - t_2)^2 n_1^2 n_2^2$$

where n_i $(i = 1, 2, 3)$ are the principal Cauchy stresses.
Show that the average of this is expressible as

$$\frac{1}{15}\left\{(t_2 - t_3)^2 + (t_3 - t_1)^2 + (t_1 - t_2)^2\right\}$$

and deduce that this expressible as

$$\frac{2}{15}\left\{[\mathbf{I_1}(\mathbf{T})]^{\mathbf{2}} - \mathbf{2I_2}(\mathbf{T})\right\}$$

where $I_1(T)$ and $I_2(T)$ are the first two principal invariants of $\mathbf{T}$
Soln.

$$t = \sigma^T n$$

now,

$$\begin{aligned} t_s &= t - (t \cdot n)n \\ t_s \cdot t_s &= [t - (t \cdot n)n] \cdot [t - (t \cdot n)n] \\ &= t \cdot t - 2(t \cdot n)(n \cdot t) + (t \cdot n)(t \cdot n) \\ &= t \cdot t - (t \cdot n)^2 \end{aligned} \tag{20}$$

If σ_1, σ_2 and σ_3 are the principal stresses

$$\sigma = \sum_{i=1}^{3} \sigma_i \, p_i \otimes p_i \qquad p_1, p_2, p_3 \text{ are principal directions} \tag{21}$$

$$n = n_i p_i$$

$$\Rightarrow t_i = \sigma_i \, n_i (\text{no repeated summation})$$

$$\Rightarrow t_s \cdot t_s = (\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2) - (\sigma_1 n_1^2 + \sigma_3 n_3^2 + \sigma_3 n_3^2) \tag{22}$$

$$n_1^2 + n_2^2 + n_3^2 = 1$$

hence,

$$\begin{aligned} \Rightarrow t_s \cdot t_s &= (\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2)(n_1^2 + n_2^2 + n_3^2) - (\sigma_1 n_1^2 + \sigma_3 n_3^2 + \sigma_3 n_3^2) \\ &= (\sigma_2 - \sigma_3)^2 n_2^2 n_3^2 + (\sigma_3 - \sigma_1)^2 n_3^2 n_1^2 + (\sigma_1 - \sigma_2)^2 n_1^2 n_2^2 \end{aligned} \tag{23}$$

now; $< n_2^2 n_3^2 >$ is obtained as follows Let,

$$\begin{aligned} n_1 &= \cos\theta && (24) \\ n_2 &= \sin\theta\cos\phi && (25) \\ n_3 &= \sin\theta\sin\phi && (26) \end{aligned}$$

$$< n_2^2 n_3^2 >= \frac{\int_0^{2\pi}\int_0^{\pi}(\sin^2\theta\sin\phi\cos\phi)^2\sin\theta d\theta d\phi}{\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi} = \frac{1}{15} \tag{27}$$

Similarly, $< n_3^2 n_1^2 >= \frac{1}{15}$ and $< n_1^2 n_2^2 >= \frac{1}{15}$

$$\begin{aligned} I_1(T) &= \sigma_1 + \sigma_2 + \sigma_3 && (28) \\ I_2(T) &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 && (29) \end{aligned}$$

upon substution

$$\frac{2}{15}\left(I_1(T)^2 - 3I_2(T)\right) = \frac{1}{15}\left((\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 + (\sigma_1-\sigma_2)^2\right) \tag{30}$$

4). Formulate the balance of angular momentum for a material body acted on by a body torque $\mathbf{c}$ per unit mass in addition to the body force $\mathbf{b}$, and a contact torque $\mathbf{u}_{(\mathbf{n})}$ per unit area in addition to the contact force $\mathbf{t}_{(\mathbf{n})}$. Establish, by a 'tetrahedron argument', the existence of a couple stress tensor μ such that $\mathbf{u}_{\mathbf{n}} = \mu^{\mathbf{T}}\,\mathbf{n}$.
Soln.
The angular momentum L is

$$L = \int_{R_t} \rho\, x \wedge v\, dR_t \tag{31}$$

$$\frac{dL}{dt} = \int_{R_r} \rho\, \frac{\partial (x \wedge v)}{\partial t} dR \tag{32}$$

since mass is conserved $\rho dR_t = \rho dR$

$$\frac{\partial (x \wedge v)}{\partial t} = v \wedge v + x \wedge a = x \wedge a \tag{33}$$

$$\Rightarrow \frac{dL}{dt} = T_{torque} = \int_{R_t} \rho\, x \wedge a\, dR_t \tag{34}$$

$$T = \int_{R_t} (-\rho b \wedge x + c) dR_t + \int_{\partial R_t} (x \wedge t + u) dA_t \tag{35}$$

thus,

$$\int_{R_t} \rho\, x \wedge a\, dR_t = \int_{R_t} (-\rho b \wedge x + c) dR_t + \int_{\partial R_t} (x \wedge t + u) dA_t$$

$$\int_{R_t} [\rho\,(x \wedge a - x \wedge b) - c] dR_t = \int_{\partial R_t} (x \wedge t + u) dA_t \tag{36}$$

but, $t = \sigma^T\, n$

$$\Rightarrow \int_{\partial R_t} (x \wedge \sigma^T n) dA_t = \int_{R_t} (x \wedge \mathrm{div}\sigma - \tau) dR_t \tag{37}$$

(Pr-20, Pg44, Chadwick)
τ is axial vector of $(\sigma - \sigma^T)$

$$\Rightarrow \int_{R_t} [x \wedge (\rho a - \rho b - \mathrm{div}\sigma) - \rho c] dR_t = \int_{R_t} -\tau dR_t + \int_{\partial R_t} u\, dA_t \tag{38}$$

using the Force balance $\rho a - \rho b - \mathrm{div}\sigma = 0$
also using arguments similar to those on 146-147 of Ogden we can show, $u = \mu^T n$ and $\int_{\partial R_t} \mu^T n dA_t = \int_{R_t} \mathrm{div}\mu dR_t$
thus,

$$\int_{R_t} (\rho c + \tau + \mathrm{div}\mu) dR_t = 0, \qquad \forall R_t \tag{39}$$

$$\Rightarrow \rho c + \tau + \mathrm{div}\mu = 0 \tag{40}$$