EN221 - Fall2008 - HW # 8 Solutions

Prof. Vivek Shenoy

1.) Consider a static situation in which a body occupies the region B_0 for all time.Assume that the body force $b = 0$ and that the body is bounded by surfaces S_0 and S_1 as shown in Fig. (a). Assume further that S_0 and S_1 are acted on by uniform pressures π_0 and π_1. Show that the average stress in B_0 is a pressure of amount

$$\frac{\pi_1 v_1 - \pi_0 v_0}{v_1 - v_0} \tag{1}$$

where v_0 and v_1 are, respectively, the volumes enclosed by S_0 and S_1. (b) Consider a steady, ir-rotational flow of an ideal fluid of density ρ_0 over an obstacle R, where R is a bounded regular region whose interior lies outside the flow region B_0 (see Fig. (b)). Assume that the body force is zero. Show that the total force exerted on R by the fluid is equal to

$$\frac{\rho_0}{2} \int_{\partial R} v^2 n \, dA \tag{2}$$

where ∂R is the boundary of R.

Soln.

(a) Please see problem 9 on pg. 105 in Chadwick.

(b) For an ideal fluid,

$$\begin{aligned}
v &= \operatorname{grad} \phi \\
\dot{v} &= \frac{\partial v}{\partial t} + \operatorname{grad} v \ \cdot v \\
&= \operatorname{grad}\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\operatorname{grad}(v^2) + (\nabla \wedge v) \wedge \ v \\
&= \operatorname{grad}\left(\frac{\partial \phi}{\partial t}\right) + frac12\operatorname{grad}(v^2)
\end{aligned}$$

The flow is steady, hence $\frac{\partial \phi}{\partial t} = 0$

$$\Rightarrow \dot{v} = \frac{1}{2}\operatorname{grad} v^2$$

Balance of linear momentum gives, note, ρ is constant,

$$\rho_o \dot{v} = -\nabla p + \rho_o b \tag{3}$$

but, there's no body force.

$$
\begin{aligned}
\Rightarrow \rho_o \dot{v} &= -\nabla p \\
\Rightarrow \operatorname{grad}\left(\rho_o \frac{v^2}{2} + p\right) &= 0 \\
\Rightarrow \rho_o \frac{v^2}{2} + p &= C(\text{constant}) \qquad (4)
\end{aligned}
$$

Thus, the force on the obstacle is

$$
\begin{aligned}
F &= -\int_{\partial R} np\, da \\
&= \int_{\partial R} \rho_o \frac{v^2}{2} n\, da - C \int_{\partial R} n\, da \\
&= \int_{\partial R} \rho_o \frac{v^2}{2} n\, da \qquad (5)
\end{aligned}
$$

$\int_{\partial R} n\, da = 0$ since, the region is closed and regular.

2.) Derive the boundary conditions (26.2c) for the flow problem sketched in Fig. 2.5 of the notes from the OCAIM institute distributed in class.
Soln.
velocity in layer-1 and 2 are given by

$$v_1 = \nabla\phi_1 \tag{6}$$
$$v_2 = \nabla\phi_2 + ue_1 \tag{7}$$

now the boundary conditions at the interface $\eta = z$ are that, any particle on the interface is given by equation

$$f(\bar{x}, t) = \eta(x_1, x_2, t) - x_3 = 0 \tag{8}$$

and that any particle on interface always stays at the interface.

$$\begin{aligned} & \frac{Df}{Dt} = 0 \\ \Rightarrow \quad & \frac{D(\eta - z)}{Dt} = 0 \\ \Rightarrow \quad & \frac{\partial(\eta - z)}{\partial t} + \text{grad}(\eta - z) \cdot v = 0 \end{aligned}$$

$$\begin{aligned} \text{grad}(\eta - z) &= \frac{\partial\eta}{\partial x_1}e_1 + \frac{\partial\eta}{\partial x_2}e_2 - e_3 \\ v &= \left(\frac{\partial\phi_1}{\partial x_1}\right)e_1 + \left(\frac{\partial\phi_1}{\partial x_2}\right)e_2 + \left(\frac{\partial\phi_1}{\partial x_3}\right)e_3 \quad \text{(layer-1)} \\ &= \left(\frac{\partial\phi_2}{\partial x_1} + u\right)e_1 + \left(\frac{\partial\phi_2}{\partial x_2}\right)e_2 + \left(\frac{\partial\phi_2}{\partial x_3}\right)e_3 \quad \text{(layer-2)} \end{aligned}$$

at the interface $\eta = x_3$

$$\begin{aligned} \frac{\partial\phi_1}{\partial x_3} &= \frac{\partial\eta}{\partial t} + \frac{\partial\phi_1}{\partial x_1}\frac{\partial\eta}{\partial x_1} + \frac{\partial\phi_1}{\partial x_2}\frac{\partial\eta}{\partial x_2} \\ \frac{\partial\phi_2}{\partial x_3} &= \frac{\partial\eta}{\partial t} + \left(\frac{\partial\phi_2}{\partial x_1} + u\right)\frac{\partial\eta}{\partial x_1} + \frac{\partial\phi_2}{\partial x_2}\frac{\partial\eta}{\partial x_2} \end{aligned}$$

picking only terms with first order derivatives,

$$\frac{\partial\phi_1}{\partial x_3} = \frac{\partial\eta}{\partial t} \tag{9}$$
$$\frac{\partial\phi_2}{\partial x_3} = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x_1} \tag{10}$$

further linearizing about $x_3 = 0$ (Eq.2.4.1 OCIAM)
we obtain

$$\frac{\partial\phi_1}{\partial x_3} = \frac{\partial\eta}{\partial t} \tag{11}$$
$$\frac{\partial\phi_2}{\partial x_3} = \frac{\partial\eta}{\partial t} + u\frac{\partial\phi_2}{\partial x_1} \tag{12}$$

Now, at the interface

$$P_1 - P_2 = -\gamma\, k \qquad \text{(Eq.2.58 OCIAM)} \tag{13}$$

Now using Bernoulli eqn for ideal incompressible ir-rotational fluid

$$\frac{\partial \phi_i}{\partial t} + \frac{1}{2}|v_i|^2 + \frac{p_i}{\rho} + g\eta = F_i(t) \tag{14}$$

where $i = 1, 2$ and $v_i = ue_1\delta_{2i} + \nabla\phi_i$ As explained in the OCIAM notes, since ϕ_i is arbitrary to a function of t, we can choose $F_i(t)$ arbitrarily.
Choosing $\rho_1 F_1(t) = \rho_2 F_2(t)$
we obtain at $\eta = x_3$

$$\rho_1\frac{\partial \phi_1}{\partial t} + \rho_1\frac{1}{2}|v_1|^2 + p_1 + g\eta\rho_1 = \rho_2\frac{\partial \phi_2}{\partial t} + \rho_2\frac{1}{2}|v_2|^2 + p_2 + g\eta\rho_2 \tag{15}$$

Using $v_i = ue_1\delta_{2i} + \nabla\phi_i$ $(i = 1, 2)$, and $p_1 - p_2 = -\gamma k$ and standard linearization in OCIAM notes, we obtain.

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + g\eta\right) - \rho_2\left(\frac{\partial \phi_2}{\partial t} + u\frac{\partial \phi_2}{\partial x_1} + g\eta\right) = \gamma\frac{\partial^2 \eta}{\partial x_1^2} \qquad (\text{at } x_3{=}0) \tag{16}$$

3). Using the traveling wave solutions (2.63 of the OCAIM notes) derive the dispersion relation (relation between ω and k, 2.64 in the notes) and verify the conditions (2.67) and (2.68) for the *Rayleigh-Taylor* and *Kelvin-Hemholtz* instabilities, respectively.
Soln.
The process for obtaining the dispersion relation is given in the mathematica work-out.
we obtain

$$\omega^2 \rho_1 \coth(kh_1) + (\omega - uk)^2 \rho_2 \coth(kh_2) = ((\rho_1 - \rho_2)g + \gamma k^2)k \qquad (17)$$

Taking limit $h_1 \to \infty$, $h_2 \to \infty$, we obtain

$$\coth(kh_1) \to 1$$

$$\coth(kh_2) \to 1$$

$$\Rightarrow \omega^2 \rho_1 + (\omega - uk)^2 \rho_2 = ((\rho_1 - \rho_2)g + \gamma k^2)k$$
$$\Rightarrow \omega^2(\rho_1 + \rho_2) - 2\omega uk\rho_2 + u^2k^2\rho_2 = ((\rho_1 - \rho_2)g + \gamma k^2)k \qquad (18)$$

for, $u = 0$, we obtain

$$\omega^2 = \frac{((\rho_1 - \rho_2)g + \gamma k^2)k}{\rho_1 + \rho_2} \qquad (k > 0) \qquad (19)$$

if $\rho_1 < \rho_2$ $(\rho_1 - \rho_2)g + \gamma k^2 < 0$

i.e., $\sqrt{\frac{(\rho_1 - \rho_2)g}{\gamma}} > k$
$\omega^2 < 0$ i.e., ω is imaginary means any disturbance will grow exponentially.

If $u \neq 0$ then Eqn(18) is a quadratic in ω. To find values of ω, for which flow is unstable we need ω to be complex.
i.e., the discriminant of the quadratic Eqn(18)< 0

$$
\begin{aligned}
& (2uk\rho_2)^2 - 4(\rho_1 + \rho_2)((\rho_1 - \rho_2)g + \gamma k^2)k + u^2k^2\rho_2) < 0 \\
\Rightarrow \quad & 4u^2k^2\rho_2^2 - 4(\rho_1 + \rho_2)((\rho_1 - \rho_2)g + \gamma k^2)k - 4(\rho_1 + \rho_2)u^2k^2\rho_2 < 0 \\
\Rightarrow \quad & u^2k^2\rho_1\rho_2 > (\rho_1 + \rho_2)\left[(\rho_1 - \rho_2)g + \gamma k^2\right]k \\
\Rightarrow \quad & u^2 > \frac{(\rho_1 + \rho_2)}{\rho_1\rho_2}\frac{\left[(\rho_1 - \rho_2)g + \gamma k^2\right]}{k} \qquad (20)
\end{aligned}
$$

4). In class we derived the Rayleigh-Plesset equation for bubble dynamics in an incompressible fluid using the Cauchy equation of motion (you can find a similar analysis in Chadwick, Problem 7, page 101). Derive this equation using the Bernoulli equation for incompressible fluids given in the class notes. Note that the flow in this case is not steady, but is ir-rotational.

References on bubble dynamics and solutions of the Rayleigh-Plesset equation:
Z. C. Feng and L. G. Leal, Non-linear bubble dynamics, Annual Reviews of Fluid Mechanics, vol. 29, pp 201-243 (1997).
Bubble Puzzles : http://www.aip.org/pt/vol-56/iss-2/p36.html
Soln.
For ir-rotational and in-compressible flow, the Bernoulli eqn becomes,

$$\text{grad}\left(\frac{\partial\phi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} = 0\right) \tag{21}$$

$v = \text{grad}\ \phi$
since there is only radial dependence
grad $\equiv \frac{\partial}{\partial r}\hat{r}$ and
$\frac{\partial\hat{r}}{\partial\theta} = 0 = \frac{\partial\hat{r}}{\partial\phi}$ so Eqn(21) $\Rightarrow$

$$\frac{\partial\phi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} = h(t) \tag{22}$$

now from the class notes,

$$\begin{aligned} v &= \text{grad}\ \phi = \frac{a^2\dot{a}}{r^2} = \frac{\partial\phi}{\partial r} \\ \phi &= -\frac{a^2\dot{a}}{r} + f(t) \end{aligned} \tag{23}$$

so;

$$\frac{\partial\phi}{\partial t} = -\frac{a^2\dot{a}}{r} - \frac{2a\dot{a}^2}{r} + f'(t) \tag{24}$$

$$\Rightarrow -\frac{a^2\dot{a}}{r} - \frac{2a\dot{a}^2}{r} + \frac{a^4\dot{a}^2}{2r^4} + \frac{P}{\rho_o} = g(t) \tag{25}$$

This is time for all r. So putting $r \to \infty$

$$\begin{aligned} \Rightarrow \frac{P(r=\infty)}{\rho_o} &= g(t) = P_\infty \quad \text{(constant)} \\ \Rightarrow P &= \rho_o\left[\frac{a^2\dot{a}}{r} + \frac{2a\dot{a}^2}{r} - \frac{a^4\dot{a}^2}{2r^4}\right] + P_\infty \end{aligned} \tag{26}$$

putting $r = a$

$$P - P_\infty = \rho_o\left[\ddot{a} + \frac{3}{2}\dot{a}\right] \tag{27}$$

also note, the Bernoulli equation 21

$$
\begin{aligned}
\frac{\partial(\text{grad }\phi)}{\partial t} + \frac{1}{2}\text{grad}\, v^2 + \frac{1}{\rho_o}\text{grad}\, P = 0 \\
\Rightarrow \frac{\partial v}{\partial t} + \text{grad}\, v \cdot v + (\nabla \wedge v) \wedge v + \frac{1}{\rho_o}\text{grad}\, P = 0
\end{aligned}
$$

Note $\nabla \wedge v = 0$
hence

$$
\Rightarrow a + \frac{1}{\rho}\text{grad}\, P = 0 \quad \text{(Equilibrium Eqn)} \tag{28}
$$

We first write down the expression for the travelling waves

```
η = A Exp[ⅈ (k x - ω t)]
f₁ = B Exp[ⅈ (k x - ω t)] Cosh[k (z + h₁)]
f₂ = C Exp[ⅈ (k x - ω t)] Cosh[k (z - h₂)]
```

$A e^{i (k x-t \omega)}$

$B e^{i (k x-t \omega)} \text{Cosh}[k (z + h_1)]$

$C e^{i (k x-t \omega)} \text{Cosh}[k (z - h_2)]$

Here, we apply the boundary conditions given in Eq.2.62c

```
E1 = (D[f₁, z] - D[η, t]) /. z → 0
E2 = (D[f₂, z] - D[η, t] - U D[η, x]) /. z → 0
E3 =
 Expand[(ρ₁ (D[f₁, t] + g η) - ρ₂ (D[f₂, t] + U D[f₂, x] + g η) - γ D[η, x, x]) /. z → 0]
EE1 = {A → 1, B → 0, C → 0}
EE2 = {A → 0, B → 1, C → 0}
EE3 = {A → 0, B → 0, C → 1}
```

$i A e^{i (k x-t \omega)} \omega + B e^{i (k x-t \omega)} k \text{Sinh}[k h_1]$

$-i A e^{i (k x-t \omega)} k U + i A e^{i (k x-t \omega)} \omega - C e^{i (k x-t \omega)} k \text{Sinh}[k h_2]$

$A e^{i (k x-t \omega)} k^2 \gamma + A e^{i (k x-t \omega)} g \rho_1 - i B e^{i (k x-t \omega)} \omega \text{Cosh}[k h_1] \rho_1 - A e^{i (k x-t \omega)} g \rho_2 - i C e^{i (k x-t \omega)} k U \text{Cosh}[k h_2] \rho_2 + i C e^{i (k x-t \omega)} \omega \text{Cosh}[k h_2] \rho_2$

$\{A \to 1, B \to 0, C \to 0\}$

$\{A \to 0, B \to 1, C \to 0\}$

$\{A \to 0, B \to 0, C \to 1\}$

Here we write the matrix M, so that M.x = 0, where x = {A, B, C}.

```
M = {{E1 /. EE1, E1 /. EE2, E1 /. EE3},
  {E2 /. EE1, E2 /. EE2, E2 /. EE3}, {E3 /. EE1, E3 /. EE2, E3 /. EE3}}
```

$\{\{i e^{i (k x-t \omega)} \omega, e^{i (k x-t \omega)} k \text{Sinh}[k h_1], 0\}, \{-i e^{i (k x-t \omega)} k U + i e^{i (k x-t \omega)} \omega, 0, -e^{i (k x-t \omega)} k \text{Sinh}[k h_2]\}, \{e^{i (k x-t \omega)} k^2 \gamma + e^{i (k x-t \omega)} g \rho_1 - e^{i (k x-t \omega)} g \rho_2, -i e^{i (k x-t \omega)} \omega \text{Cosh}[k h_1] \rho_1, -i e^{i (k x-t \omega)} k U \text{Cosh}[k h_2] \rho_2 + i e^{i (k x-t \omega)} \omega \text{Cosh}[k h_2] \rho_2\}\}$

If we want non−trivial values of A, B, C, the determinant of matrix M should be zero.

```
FullSimplify[Det[M],
 Assumptions → {k > 0, h₁ > 0, h₂ > 0, ω > 0, g > 0, ρ₁ > 0, ρ₂ > 0, U > 0}]
```

$-e^{3 i (k x-t \omega)} k (-(\omega^2 \text{Cosh}[k h_1] - g k \text{Sinh}[k h_1]) \text{Sinh}[k h_2] \rho_1 + \text{Sinh}[k h_1] (-(-k U + \omega)^2 \text{Cosh}[k h_2] \rho_2 + k \text{Sinh}[k h_2] (k^2 \gamma - g \rho_2)))$

This implies that

$$(\omega^2 \text{Cosh}[k\, h_1] - g\, k\, \text{Sinh}[k\, h_1])\, \text{Sinh}[k\, h_2]\, \rho_1 + \text{Sinh}[k\, h_1]\, (-(-k\, U + \omega)^2\, \text{Cosh}[k\, h_2]\, \rho_2 + k\, \text{Sinh}[k\, h_2]\, (k^2\, \gamma - g\, \rho_2)) = 0$$

After some rearrangements, we will recover Eq.2.64