EN221 - Fall2008 - HW # 9 Solutions

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1.) Consider the observer transformation discussed in class is defined by the relation

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x} \tag{1}$$

where $\mathbf{Q}$ is an orthogonal tensor. (a) Show that

$$\mathrm{div}_{x^*}\mathbf{T}^*(\mathbf{x}^*, t) = \mathbf{Q}(t)\mathrm{div}_x\mathbf{T}(\mathbf{x}, t) \tag{2}$$

where $\mathbf{T}$ is Cauchy stress
(b) Show that the acceleration transforms according to

$$\dot{\mathbf{v}}^*(\mathbf{x}^*, t) = \mathbf{Q}(t)\dot{\mathbf{v}}(\mathbf{x}, t) + \ddot{\mathbf{c}}(t) + 2\dot{\mathbf{Q}}(t)\mathbf{v}(\mathbf{x}, t) + \ddot{\mathbf{Q}}(t)\mathbf{x} \tag{3}$$

(c) Show further that if the body force transforms according to

$$\mathbf{b}^*(\mathbf{x}^*, t) = \mathbf{Q}(t)\mathbf{b}(\mathbf{x}, t) \tag{4}$$

then the equation of motion is given by

$$\mathrm{div}_{\mathbf{x}^*}\mathbf{T}^* + \rho^*\mathbf{b}^* + \mathbf{k} = \rho^*\dot{\mathbf{v}}^* \tag{5}$$

where $\rho^*(\mathbf{x}^*) = \rho(\mathbf{x}, t)$ and determine a relation for $\mathbf{k}$
(d) because of the additional term $\mathbf{k}$, the equation of motion is not *not invariant* under all changes in observer. Show that $\mathbf{k}$ vanishes for observer for whom $\mathbf{Q}$ and $\dot{\mathbf{c}}$ are constant. Such observer are called *Galilean* and have the property of being accelerationless with respect to the underlying inertial observer.

Soln.
The observer transformation is

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}$$

(a)

$$\mathrm{div}_{\mathbf{x}^*}\mathbf{T}^* = \frac{\partial T_{ij}}{\partial x_i^*}e_j^* \tag{6}$$

$$\begin{aligned} T_{ij}^*(\mathbf{x}^*, t) &= T_{ij}(\mathbf{x}, t) \\ \frac{T_{ij}^*(\mathbf{x}^*, t)}{\partial \mathbf{x}} &= \frac{\partial T_{ij}(\mathbf{x}, t)}{\partial \mathbf{x}} \end{aligned}$$

$$
\begin{aligned}
\mathrm{div}_{\mathbf{x}^*}\mathbf{T}^* &= \frac{\partial T_{ij}}{\partial x_i^*} e_j^* \\
&= \frac{\partial T_{ij}}{\partial x_i}\mathbf{Q}(t)e_j \\
&= \mathbf{Q}(t)\frac{\partial T_{ij}}{\partial x_i} e_j \\
&= \mathbf{Q}(t)\mathrm{div}_{\mathbf{x}}\mathbf{T}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathrm{div}_{\mathbf{x}^*}\mathbf{T}^* &= \mathbf{Q}(t)\mathrm{div}_{\mathbf{x}}\mathbf{T} \\
&= \mathbf{Q}(t)(-\rho\mathbf{b} + \rho\dot{\mathbf{v}}) \\
&= -\rho\mathbf{Q}(t)\mathbf{b} + \rho\mathbf{Q}(t)\dot{\mathbf{v}}
\end{aligned}
$$

But,

$$
\begin{aligned}
\mathbf{x}^* &= \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x} \\
\dot{\mathbf{x}}^* = \mathbf{v}^* &= \dot{\mathbf{c}}(t) + \mathbf{Q}(t)\dot{\mathbf{x}} + \dot{\mathbf{Q}}(t)\mathbf{x} \\
\ddot{\mathbf{x}}^* = \dot{\mathbf{v}}^* &= \ddot{\mathbf{c}}(t) + \mathbf{Q}(t)\ddot{\mathbf{x}} + \ddot{\mathbf{Q}}(t)\mathbf{x} + 2\dot{\mathbf{Q}}(t)\dot{\mathbf{x}} \\
&= \ddot{\mathbf{c}}(t) + \mathbf{Q}(t)\dot{\mathbf{v}} + \ddot{\mathbf{Q}}(t)\mathbf{x} + 2\dot{\mathbf{Q}}(t)\mathbf{v}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathrm{div}_{\mathbf{x}^*}\mathbf{T}^* &= -\rho\mathbf{Q}(t)\mathbf{b} + \rho\mathbf{Q}(t)\dot{\mathbf{v}} && (7) \\
&= -\rho\mathbf{Q}(t)\mathbf{b} + \rho\left[\dot{\mathbf{v}}^* - \ddot{\mathbf{c}}(t) - \ddot{\mathbf{Q}}(t)\mathbf{x} - 2\dot{\mathbf{Q}}(t)\mathbf{v}\right] && (8)
\end{aligned}
$$

using $\rho^* = \rho$

$$
\mathrm{div}_{\mathbf{x}^*}\mathbf{T}^* + \rho^*\mathbf{b}^* + \rho^*\left[\ddot{\mathbf{c}}(t) + \ddot{\mathbf{Q}}(t)\mathbf{x} + 2\dot{\mathbf{Q}}(t)\mathbf{v}\right] = \rho^*\dot{\mathbf{v}}^* \qquad (9)
$$

where Internal body force in the accelerating frame is
$\mathbf{k} = \ddot{\mathbf{c}}(t) + \ddot{\mathbf{Q}}(t)\mathbf{x} + 2\dot{\mathbf{Q}}(t)\mathbf{v}$ if $\mathbf{Q}$ and $\mathbf{c}$ are constant then $\mathbf{k} = 0$

2.) The Transformation rule for the Cauchy stress tensor $\mathbf{T}$ for relative motion of observers is

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} \tag{10}$$

Tensors that transform in this manner are called objective tensors. Is the material rate of change (or material derivative) of Cauchy stress $\dot{\mathbf{T}}$ objective ?, i.e., is $\dot{\mathbf{T}}^* = \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}}$?

(b) Show that if the Cauchy stress tensor is objective, then the *Jaumann stress rate* defined by

$$\breve{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W} \tag{11}$$

is objective. Note that $\mathbf{W}$ in the above equation represents the spin tensor, $\frac{1}{2}(\mathbf{L} - \mathbf{L}^{\mathrm{T}})$. This stress rate is also called the *co-rotational rate* of Cauchy stress. Why is this a reasonable name for this stress rate?

Soln.

(a)

$$\begin{aligned} \mathbf{T}^* &= \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} \\ \dot{\mathbf{T}}^* &= \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}} \end{aligned} \tag{12}$$

Additional terms $\dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}}$, so $\dot{\mathbf{T}}$ is not objective

(b)

$$\begin{aligned} \breve{\mathbf{T}} &= \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W} \\ \breve{\mathbf{T}}^* &= \dot{\mathbf{T}}^* - \mathbf{W}^*\mathbf{T}^* + \mathbf{T}^*\mathbf{W}^* \end{aligned} \tag{13}$$

$$\boldsymbol{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} = -\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}} \tag{14}$$

$$\begin{aligned} \mathbf{W}^* &= \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}} + \boldsymbol{\Omega} \\ \mathbf{W}^* &= \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} \end{aligned} \tag{15}$$

Substituting Eqn(12 and 15) in Eqn(13)

$$\begin{aligned} \breve{\mathbf{T}}^* &= \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}}\mathbf{T}^* - \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\mathbf{T}^* + \mathbf{T}^*\mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}} - \mathbf{T}^*\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}} \\ &= \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} - \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}} \\ &= \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{W}\mathbf{T}\mathbf{Q}^{\mathrm{T}} - \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\mathbf{W}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}} \\ &= \mathbf{Q}(\dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W})\mathbf{Q}^{\mathrm{T}} \end{aligned} \tag{16}$$

Thus Jaumann stress rate is frame independent.

This is the rate of change of $\mathbf{T}$ relative to a basis rotating with the local body spin $\boldsymbol{\Omega}$.

3). For each of the following constitutive equations decide whether or not the principle of objectivity is satisfied. (α and *beta* are scalar constants, p a scalar valued function and $\mathbf{f}$ a symmetric tensor-valued function.)

(i) $\sigma = -p(t)\mathbf{I}$

(ii) $\sigma = \alpha(\mathbf{F} + \mathbf{F}^{\mathbf{T}})$

(iii) $\sigma = \mathbf{f}(\mathbf{v})$

(iv) $\sigma = \alpha\{\text{grad } \mathbf{a} + (\text{grad } \mathbf{a})^{T} + 2\mathbf{L}^{\mathbf{T}}\mathbf{L}\}$

(v) $\sigma = \mathbf{f}(\mathbf{b})$

(vi) $\dot{\sigma} = \mathbf{W}\sigma - \sigma\mathbf{W} + (\alpha \, \text{tr } \mathbf{D})\mathbf{I} + \beta\mathbf{D}$

Soln.

(i)

$$\begin{aligned} \sigma &= -p(t)\mathbf{I} \\ \sigma^*(x^*, t^*) &= -p(t^*)I \\ &= -p(t - t_o^*)I \end{aligned} \tag{17}$$

$t^* = t - t_o^*$; $t = t_o^*$ is origin for t^* (There is some origin in time $t_o(t_o^*) = 0$) but, $t - t_o^* = t$

$\Rightarrow \sigma(x^*, t^*) = -p(t)I = \sigma(x, t)$

(ii)

$$\begin{aligned} \sigma &= \alpha(\mathbf{F} + \mathbf{F}^{\mathbf{T}}) \\ \sigma^* &= \alpha(\mathbf{F}^* + \mathbf{F}^{*\mathbf{T}}) \end{aligned}$$

$$\begin{aligned} F^* &= *QF \\ \Rightarrow \sigma^* &= \alpha(\mathbf{QF} + \mathbf{F}^{\mathbf{T}}\mathbf{Q}^{\mathbf{T}}) \\ &= \alpha\mathbf{Q}(\mathbf{F} + \mathbf{F}^{\mathbf{T}})\mathbf{Q}^{\mathbf{T}} - \alpha\mathbf{QFQ}^{\mathbf{T}} - \mathbf{Q} \end{aligned}$$

so, σ^* is not objective

(iii)

$$\begin{aligned} \sigma &= \mathbf{f}(\mathbf{v}) \\ \sigma^* &= \mathbf{f}(\mathbf{v}^*) \\ &= \mathbf{f}(\dot{\mathbf{c}} + \dot{\mathbf{Q}}\mathbf{x} + \mathbf{Q}\mathbf{v}) \\ &= \mathbf{Q}\mathbf{f}(\mathbf{v})\mathbf{Q}^{\mathbf{T}} \quad \text{for the relation to be objectivet} \end{aligned} \tag{18}$$

$$\Leftrightarrow \mathbf{f}(\dot{\mathbf{c}} + \dot{\mathbf{Q}}\mathbf{x} + \mathbf{Q}\mathbf{v}) = \mathbf{Q}\mathbf{f}(\mathbf{v})\mathbf{Q}^{\mathbf{T}} \tag{19}$$

putting $\mathbf{v} = 0$ and $\mathbf{Q} = \mathbf{I}$

$$\mathbf{f}(\dot{\mathbf{c}}) = \mathbf{f}(\mathbf{0}) \qquad \forall\dot{\mathbf{c}} \tag{20}$$

$\Rightarrow \mathbf{f(v)}$ is a constant$= C$

$$\mathbf{f}(\dot{\mathbf{c}} + \dot{\mathbf{Q}}\mathbf{x} + \mathbf{Q}\mathbf{v}) = \mathbf{Q}\mathbf{f}(\mathbf{v})\mathbf{Q}^{\mathbf{T}} \tag{21}$$

putting $\mathbf{c} = 0$ and $\dot{\mathbf{Q}} = \mathbf{0}$

$$\Rightarrow \mathbf{f}(\mathbf{Q}\mathbf{v}) = \mathbf{C} = \mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathbf{T}} \quad \forall \mathbf{Q} \tag{22}$$

putting a few values of $\mathbf{Q}$ we will obtain
$C = pI$; wher p is a constant is a necessary and sufficient for objectivity
(iv)

$$\sigma = \alpha\{\text{grad}\,\mathbf{a} + (\text{grad}\,\mathbf{a})^T + 2\mathbf{L}^{\mathbf{T}}\mathbf{L}\} \tag{23}$$

Let $A_2 = \text{grad}\,\mathbf{a} + (\text{grad}\,\mathbf{a})^T + 2\mathbf{L}^{\mathbf{T}}\mathbf{L}$ from problem 4.1 Chadwick
From the hint given on pg.169 and problem 4.1

$$\begin{aligned}
A_2 &= \mathring{\mathbf{A}}_1 \qquad (\mathbf{A_1} = \mathbf{2D}) \\
\mathring{\mathbf{A}}_\mathbf{1} &= \dot{\mathbf{A}}_\mathbf{1} + \mathbf{L}^{\mathbf{T}}\mathbf{A_1} + \mathbf{A_1}\mathbf{L} \\
&= \mathbf{2}\frac{\left(\dot{\mathbf{L}} + \dot{\mathbf{L}}^{\mathbf{T}}\right)}{\mathbf{2}} + \mathbf{L}^{\mathbf{T}}\mathbf{2}\frac{\left(\mathbf{L} + \mathbf{L}^{\mathbf{T}}\right)}{\mathbf{2}} + \mathbf{2}\frac{\left(\mathbf{L} + \mathbf{L}^{\mathbf{T}}\right)}{\mathbf{2}}\mathbf{L} \\
&= \dot{\mathbf{L}} + \dot{\mathbf{L}}^{\mathbf{T}} + \mathbf{L^2} + \mathbf{L^{2^T}} + \mathbf{2L^T L} \qquad (24) \\
\Rightarrow \sigma &= \alpha\mathbf{A_2} \qquad (25)
\end{aligned}$$

$\mathbf{A_1}$ is $\mathbf{2D}$ and clearly objective (pg. 134 Chadwick)
$\mathbf{A_2} = \mathring{\mathbf{A}}_\mathbf{1}$
and from problem 4(pg.134)
Since $\mathbf{A_1}$ is objective $\mathbf{A_2}$ is objective
Thus $\sigma = \alpha\mathbf{A_2}$ is objective

(v)

$$\sigma = \mathbf{f}(\mathbf{b}) \tag{26}$$

Using Problem.1 of the HW

$$\begin{aligned}
\mathbf{b}^* &= \mathbf{b} + (\ddot{\mathbf{c}} + \mathbf{2}\dot{\mathbf{Q}}\mathbf{v} + \ddot{\mathbf{Q}}\mathbf{x}) \\
\Rightarrow \sigma^* &= \mathbf{f}(\mathbf{b}^*) \\
&= \mathbf{f}(\mathbf{b} + \ddot{\mathbf{c}} + \mathbf{2}\dot{\mathbf{Q}}\mathbf{v} + \ddot{\mathbf{Q}}\mathbf{x}) \qquad (27) \\
\sigma^* &= \mathbf{Q}\sigma\mathbf{Q}^{\mathbf{T}} \\
& \qquad (28)
\end{aligned}$$

$$\Leftrightarrow \mathbf{f}(\mathbf{b} + \ddot{\mathbf{c}} + \mathbf{2}\dot{\mathbf{Q}}\mathbf{v} + \ddot{\mathbf{Q}}\mathbf{x}) = \mathbf{Q}\mathbf{f}(\mathbf{b})\mathbf{Q}^{\mathbf{T}} \tag{29}$$

Making $\dot{\mathbf{Q}} = 0$, $\ddot{\mathbf{Q}} = 0$, $\mathbf{b} = 0$, $\mathbf{Q} = \mathbf{I}$

$$\mathbf{f}(\ddot{\mathbf{c}}) = \mathbf{Q}\mathbf{f}(\mathbf{0})\mathbf{Q}^{\mathbf{T}} = \mathbf{f}(\mathbf{0}) \quad \forall \mathbf{c} \tag{30}$$

Thus, $\mathbf{f}(\ddot{\mathbf{c}}) = \mathbf{f}(\mathbf{0}) = \mathbf{C}(\text{constant})$ Putting $\ddot{\mathbf{c}} = \dot{\mathbf{Q}} = \ddot{\mathbf{Q}} = \mathbf{0}$

$$\mathbf{f}(\mathbf{b}) = \mathbf{C} = \mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathbf{T}} \tag{31}$$

Thus, by the same argument as in (iii) $c = bI$, where b is some constant, is necessary and sufficient for objectivity.

(vi)

$$\begin{aligned}
\dot{\sigma} &= \mathbf{W}\sigma - \sigma\mathbf{W} + (\alpha \,\mathrm{tr}\, \mathbf{D})\mathbf{I} + \beta\mathbf{D} \\
\dot{\sigma^*} &= \mathbf{W}^*\sigma^* - \sigma^*\mathbf{W}^* + (\alpha \,\mathrm{tr}\, \mathbf{D}^*)\mathbf{I} + \beta\mathbf{D}^* \\
\mathbf{W}^* &= \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathbf{T}} + \mathbf{\Omega} \\
\mathbf{\Omega} &= \dot{\mathbf{Q}}\mathbf{Q}^{\mathbf{T}} \\
\sigma^* &= \mathbf{Q}\sigma\mathbf{Q}^{\mathbf{T}} \\
\mathbf{D}^* &= \mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathbf{T}} \\
\Rightarrow \mathrm{tr}\, \mathbf{D}^* &= \mathrm{tr}\, (\mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathbf{T}}) = \mathrm{tr}\, (\mathbf{Q}^{\mathbf{T}}\mathbf{Q}\mathbf{D}) = \mathrm{tr}\, \mathbf{D}
\end{aligned} \tag{32}$$

$$\begin{aligned}
\dot{\sigma}^* &= (\mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathbf{T}} + \mathbf{\Omega})\mathbf{Q}\sigma\mathbf{Q}^{\mathbf{T}} - \mathbf{Q}\sigma\mathbf{Q}^{\mathbf{T}}(\mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathbf{T}} + \mathbf{\Omega}) + \alpha \,\mathrm{tr}\, \mathbf{D} + \beta\mathbf{Q}\sigma\mathbf{Q}^{\mathbf{T}} \\
&= \mathbf{Q}\mathbf{W}\sigma\mathbf{Q}^{\mathbf{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathbf{T}}\mathbf{Q}\sigma\mathbf{Q}^{\mathbf{T}} - \mathbf{Q}\sigma\mathbf{W}\mathbf{Q}^{\mathbf{T}} + \mathbf{Q}\sigma\mathbf{Q}^{\mathbf{T}}\mathbf{Q}\dot{\mathbf{Q}}^{\mathbf{T}} + \mathbf{Q}^{\mathbf{T}}\alpha \,\mathrm{tr}\, (\mathbf{D})\mathbf{I}\mathbf{Q}^{\mathbf{T}} + \beta(\mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathbf{T}}) \\
&= \mathbf{Q}(\mathbf{W}\sigma + \sigma\mathbf{W} + \alpha \,\mathrm{tr}\, (\mathbf{D})\mathbf{I} + \beta\mathbf{D})\mathbf{Q}^{\mathbf{T}} + \dot{\mathbf{Q}}\sigma\mathbf{Q}^{\mathbf{T}} + \mathbf{Q}\sigma\dot{\mathbf{Q}}^{\mathbf{T}} \\
&= \mathbf{Q}\dot{\sigma}\mathbf{Q}^{\mathbf{T}} + \dot{\mathbf{Q}}\sigma\mathbf{Q}^{\mathbf{T}} + \mathbf{Q}\sigma\dot{\mathbf{Q}}^{\mathbf{T}} \\
&= (\mathbf{Q}\sigma\mathbf{Q}^{\mathbf{T}})^{\dot{}}
\end{aligned} \tag{33}$$

$\Rightarrow \dot{\sigma}$ is objective.

4).The stress response of a certain type of material which exhibits both elastic and viscous properties is described by the consecutive equation

$$\sigma = \mathbf{f}(\mathbf{F}, \dot{\mathbf{F}}) \tag{34}$$

$\mathbf{f}$ being a symmetric tensor-valued function. Investigate the restriction imposed on $\mathbf{f}$ by the principle of objectivity and hence show that the most general form of Eqn(34) is

$$\begin{aligned} \mathbf{F}^* &= \mathbf{QF} \\ \Rightarrow \sigma^* &= \mathbf{f}(\mathbf{F}^*, \dot{\mathbf{F}}^*) \\ &= \mathbf{Qf}(\mathbf{F}, \dot{\mathbf{F}})\mathbf{Q}^{\mathbf{T}} \qquad \text{(for objectivity)} \\ \Rightarrow \mathbf{f}(\mathbf{QF}, (\dot{\mathbf{QF}})) &= \mathbf{Qf}(\mathbf{F}, \dot{\mathbf{F}})\mathbf{Q}^{\mathbf{T}} \end{aligned} \tag{35}$$

writing $\mathbf{F} = \mathbf{RU}$, we get

$$\begin{aligned} \mathbf{f}(\mathbf{QRU}, (\dot{\mathbf{QRU}})) &= \mathbf{Qf}(\mathbf{F}, \dot{\mathbf{F}})\mathbf{Q}^{\mathbf{T}} \\ (\dot{\mathbf{QRU}}) &= \dot{\mathbf{Q}}\mathbf{RU} + \mathbf{Q}\dot{\mathbf{R}}\mathbf{U} + \mathbf{QR}\dot{\mathbf{U}} \end{aligned} \tag{36}$$

Since $\mathbf{Q}$ is arbitrary, on putting $\mathbf{Q} = \mathbf{R}^{\mathbf{T}}$

$$\begin{aligned} (\dot{\mathbf{QRU}}) &= \dot{\mathbf{R}}^{\mathbf{T}}\mathbf{RU} + \mathbf{R}^{\mathbf{T}}\dot{\mathbf{R}}\mathbf{U} + \mathbf{R}^{\mathbf{T}}\mathbf{R}\dot{\mathbf{U}} \\ &= (\dot{\mathbf{R}}^{\mathbf{T}}\mathbf{R} + \mathbf{R}^{\mathbf{T}}\dot{\mathbf{R}})\mathbf{U} + \dot{\mathbf{U}} \end{aligned}$$

but $\mathbf{R}^{\mathbf{T}}\mathbf{R} = \mathbf{I}$ and
$\dot{\mathbf{R}}^{\mathbf{T}}\mathbf{R} + \mathbf{R}^{\mathbf{T}}\dot{\mathbf{R}} = \mathbf{0}$
hence

$$\begin{aligned} (\dot{\mathbf{QRU}}) &= \dot{\mathbf{U}} \\ \Rightarrow \mathbf{f}(\mathbf{U}, \dot{\mathbf{U}}) &= \mathbf{Qf}(\mathbf{F}, \dot{\mathbf{F}})\mathbf{Q}^{\mathbf{T}} = \mathbf{R}^{\mathbf{T}}\mathbf{f}(\mathbf{F}, \dot{\mathbf{F}})\mathbf{R} \\ \Rightarrow \mathbf{f}(\mathbf{U}, \dot{\mathbf{U}}) &= \sigma = \mathbf{R}^{\mathbf{T}}\mathbf{f}(\mathbf{U}, \dot{\mathbf{U}})\mathbf{R} \end{aligned} \tag{37}$$