EN221 HW#10

1 $$W = \sum_{n=1}^{N} \mu_n(\lambda_1^n + \lambda_2^n + \lambda_3^n)/n$$

$$F = I + \gamma e_1 \otimes e_2$$

$$\Rightarrow FF^T = (I + \gamma e_1 \otimes e_2)(I + \gamma e_2 \otimes e_1)$$

$$B = I + \gamma e_1 \otimes e_2 + \gamma e_2 \otimes e_1 + \gamma^2 e_1 \otimes e_1$$

$$B = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ~~Note~~

Eigenvalues of B are (Using Mathematica)

$$\lambda_1^2 = 1 \Rightarrow \lambda_1 = 1$$

$$\lambda_2^2 = \frac{1}{2}\left(2+\gamma^2 - \gamma\sqrt{4+\gamma^2}\right) = \cancel{4}\left[\frac{1}{2}\left(\gamma - \sqrt{4+\gamma^2}\right)\right]^2$$

$$\Rightarrow \lambda_2 = -\frac{1}{2}\left(\gamma - \sqrt{4+\gamma^2}\right)$$ (note, the $-$ve sign to make $\lambda_2 > 0$)

Similarly,

$$\lambda_3 = \frac{1}{2}\left(\gamma + \sqrt{4+\gamma^2}\right)$$ [Note that ~~λ~~ $\gamma = \frac{1}{\lambda} + \lambda$ has λ_1 and λ_2 as solution)

Eigenvectors are, $p_1 = (0, 0, 1)^T$

$$p_2 = \left(\frac{-\lambda_2}{\sqrt{1+\lambda_2^2}}, \frac{1}{\sqrt{1+\lambda_2^2}}, 0\right)^T$$

$$p_3 = \left(\frac{\lambda_3}{\sqrt{1+\lambda_3^2}}, \frac{1}{\sqrt{1+\lambda_3^2}}, 0\right)^T$$

also note; $$\frac{1}{\lambda_3} = \frac{1}{\frac{\gamma + \sqrt{4+\gamma^2}}{2}} = \frac{\gamma - \sqrt{4+\gamma^2}}{\frac{\gamma^2 - 4 - \gamma^2}{2}} = -\frac{\left(\gamma - \sqrt{4+\gamma^2}\right)}{2} = \lambda_2$$

$$\Rightarrow \lambda_3 = \frac{1}{\lambda_2};$$

$$\sigma = \frac{1}{J}\sum_{i=1}^{3} \lambda_i \frac{\partial W}{\partial \lambda_i} e_i \otimes e_i$$ (e_i as principal vectors of V)

(Since $J = 1$ we can drop it and add a $-pI$ term)

Now,

$$W = \sum_{i=1}^{3} \sum_{n=1}^{N} \frac{\mu_n}{n} \lambda_i^{n}$$

$$\Rightarrow \sigma = \sum_{i=1}^{3} \lambda_i \frac{\partial W}{\partial \lambda_i}\, p_i \otimes p_i \quad - p$$

$$= \sum_{i=1}^{3} \left[\sum_{n} \mu_n \lambda_i \lambda_i^{n-1} \right] p_i \otimes p_i$$

now; $\sigma_{12} = e_1 \cdot \sigma\, e_2$

$$= \sum_i \sum_n \mu_n \lambda_i^{n} (e_1 \cdot p_i)(e_2 \cdot p_i)$$

$$= \sum_n \mu_n \left[\sum_i \lambda_i^{n} (e_1 \cdot p_i)(e_2 \cdot p_i) \right]$$

$$= \sum_n \mu_n \left[\lambda_1^{n} (e_1 \cdot p_1)(e_2 \cdot p_1) + \lambda_2^{n} (e_1 \cdot p_2)(e_2 \cdot p_2) + \lambda_3^{n} (e_1 \cdot p_3)(e_2 \cdot p_3) \right]$$

$$= \sum_n \mu_n \left[\lambda_1^{n}\, 0 + \lambda_2^{n} \frac{\lambda_2}{(\sqrt{1+\lambda_2^2})(\sqrt{1+\lambda_2^2})} + \lambda_3^{n} \frac{(-\lambda_3)}{\sqrt{1+\lambda_3^2}\,\sqrt{1+\lambda_3^2}} \right]$$

$$= \sum_n \mu_n \left[\frac{\lambda_2^{n} \lambda_2}{(1+\lambda_2^2)} + \frac{\lambda_3^{n}(-\lambda_3)}{(1+\lambda_3^2)} \right]$$

But $\lambda_3 = \frac{1}{\lambda_2} \Rightarrow$ (

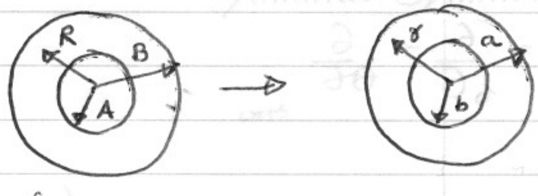
$$\sigma_{12} = \sum_n \mu_n \left[\frac{\lambda^{n} \lambda}{1+\lambda^2} - \frac{1}{\lambda^{n}} \left(\frac{1/\lambda}{1+(1/\lambda)^2} \right) \right] \quad \left(\text{writing } \lambda_3 = \frac{1}{\lambda}\right)$$

$$= \frac{\lambda}{1+\lambda^2} \sum_n \mu_n \left(\lambda^{n} - \frac{1}{\lambda^{n}} \right)$$

$$\Rightarrow \left(\lambda + \frac{1}{\lambda} \right) \sigma_{12} = \sum_n \mu_n \left(\lambda^{n} - \frac{1}{\lambda^{n}} \right) \quad //$$

2) The mapping for the problem in cylindrical co-ordinates is as follows:

$$R \rightarrow r$$
$$\Theta \rightarrow \theta$$
$$z \rightarrow -z$$



By conservation of volume (per unit length in z-direction)

$$\pi(R^2 - A^2) = \pi(a^2 - r^2)$$
$$\Rightarrow \quad R^2 = A^2 + a^2 - r^2$$
$$\Rightarrow \quad R = \sqrt{A^2 + a^2 - r^2} \quad \left(r = \left(-R^2 + A^2 + a^2\right)^{1/2}\right)$$

The deformation gradient is

$$\bar{r} = r e_r + z e_z = r E_R - z E_z \quad \left(\begin{matrix} e_r = E_R \\ e_z = E_z \end{matrix}\right)$$

$$F = \nabla \otimes r = \frac{\partial r E_R}{\partial R} \otimes E_R + \frac{1}{R}\frac{\partial r E_R}{\partial \Theta} \otimes E_\Theta - \frac{\partial z E_z}{\partial z} \otimes E_z$$

$$= \frac{\partial r}{\partial R} E_R \otimes E_R + \frac{r}{R} E_\Theta \otimes E_\Theta - 1 E_z \otimes E_z$$

$$= \lambda_R E_R \otimes E_R + \lambda_\Theta E_\Theta \otimes E_\Theta + \lambda_3 E_z \otimes E_z$$

Note that; $\lambda_R \lambda_\Theta \lambda_3 = 1$ (Can be shown by Simple calculation)

$$FF^T = \lambda_R^2 E_R \otimes E_R + \lambda_\Theta^2 E_\Theta \otimes E_\Theta + \lambda_3^2 E_z \otimes E_z$$

$$\cancel{FF}^T = \quad W = \mu/2\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \quad \begin{matrix} \lambda_1 = \lambda_R \\ \lambda_2 = \lambda_\Theta \\ \lambda_3 = \lambda_z = +1 \end{matrix}$$

$$\Rightarrow \sigma = \lambda_R \frac{\partial W}{\partial \lambda_R} E_R \otimes E_R + \lambda_\Theta \frac{\partial W}{\partial \lambda_\Theta} E_\Theta \otimes E_\Theta$$

$$+ \lambda_z \frac{\partial W}{\partial \lambda_z} E_z \otimes E_z - pI$$

$$= \mu \lambda_R^2 E_R \otimes E_R + \mu \lambda_\Theta^2 E_\Theta \otimes E_\Theta + \mu \lambda_z^2 E_z \otimes E_z - pI$$

$$= (\mu \lambda_R^2 - p) E_R \otimes E_R + (\mu \lambda_\Theta^2 - p) E_\Theta \otimes E_\Theta + (\mu \lambda_z^2 - p) E_z \otimes E_z$$

Using, $\operatorname{div}\sigma = 0$,

$\Rightarrow$ in, e_θ direction $\Rightarrow \dfrac{\partial p}{\partial \theta} = 0$

in, e_z direction $\Rightarrow \dfrac{\partial p}{\partial z} = 0 \Rightarrow p = f(r)$.

So everything is a function of r.

Writing Equation in the radial direction, (~~also~~ note that by cylindrical symmetry)

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial z} = 0$$

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\Rightarrow \frac{d\sigma_{rr}}{dr} = \mu\left(\frac{\lambda_\theta^2 - \lambda_R^2}{r}\right) \cancel{\lambda_\theta \quad \mu\lambda_R - p}$$

$$\Rightarrow \int_a^b \frac{d\sigma_{rr}}{\cancel{r}} = \int_a^b \mu\frac{(\lambda_\theta^2 - \lambda_R^2)}{r}\,dr$$

$$\sigma_{rr}\Big|_a^b = \int_a^b \mu\frac{(\lambda_\theta^2 - \lambda_R^2)}{r}\,dr \qquad \left(\text{but } \begin{array}{l}\sigma_{rr}(a) = 0 \\ \sigma_{rr}(b) = 0\end{array}\right)$$

$$\Rightarrow 0 = \int_a^b \mu\frac{(\lambda_\theta^2 - \lambda_R^2)}{r}\,dr;$$

also; $b = \sqrt{A^2 + a^2 - B^2}$ (volume cons)

$$\Rightarrow 0 = \int_a^{\sqrt{A^2+a^2-B^2}} \mu\left(\frac{\lambda_\theta^2(a,r) - \lambda_R^2(a,r)}{r}\right)dr$$

Thus we have a equation in terms of A, B, a, which could be solved to give a, and hence, b.

(b) $$\int_a^r d\sigma_{rr} = \underbrace{\int_a^r \mu\frac{(\lambda_\theta^2 - \lambda_R^2)}{r'}\,dr'}_{} = \sigma_{rr}(r) - 0 = \sigma_{rr}$$

So; $\sigma_{rr} = f(A, B, \mu)$ (Note, by Symmetry $\cancel{\frac{\partial p}{\partial\theta} = 0}$

$$\sigma_{rr} = \mu\lambda_R^2 - p = f(A, B, \mu)$$

$$\Rightarrow p(r) = \mu\lambda_R^2(r, A, B) - f(A, B, \mu)$$

$$\sigma_{\theta\theta} = \mu\lambda_\theta^2(r, A, B) - p(r) = \mu(\lambda_\theta^2 - \lambda_R^2) + f$$

$$\sigma_{zz} = \mu\underset{1}{\underbrace{\lambda_z^2}} - p(r) = \mu(1 - \lambda_R^2) + f.$$

3) The ball of radius A has homogenous deformation

$\Rightarrow$ $\cancel{\lambda_R = \lambda_\theta \neq \lambda_z \neq}$

F = constant.

now, $\lambda_R = \dfrac{dr}{dR} = \alpha$ (const)

$\Rightarrow \quad r = \alpha R + \cancel{\beta}^{\,0}. \quad (r(0)=0)$

$\Rightarrow \quad r = \alpha R$

$\Rightarrow$ By symmetry $\quad \lambda_\theta = \lambda_\phi = \sqrt{\dfrac{4\pi r^2}{4\pi R^2}} = \alpha = \dfrac{1}{\gamma}$

$\Rightarrow B = FF^T = \alpha^2 I = \dfrac{1}{\gamma^2} I$

$I_B = \dfrac{3}{\gamma^2}$

$II_B = \dfrac{3}{\gamma^4}$

$III_B = \dfrac{1}{\gamma^6}.$

$$\sigma = III_B^{-3/2}\left[\left\{\psi(III_B) - \beta II_B\right] I + \left\{\alpha III_B + \beta I_B\right\} B - \beta B^2\right]$$

We know from P14 (p.158)

$$\psi(III_B) = -\alpha III_B^{(1-3\nu)/(1-2\nu)} + \beta III_B^{(1-\nu)/(1-2\nu)}$$

for $\nu = 1/2$

$$\sigma = III_B^{-3/2}\left[\left\{-\alpha III_B^{1/2} + \beta III_B^{3/2} - \beta II_B\right\} I + (\alpha III_B + \beta I_B) B - \beta B^2\right]$$

$\cancel{\sigma}$ p

$$\sigma = \frac{\gamma^9}{\cancel{\gamma^6}}\left[\left(-\alpha\gamma^{-3} + \beta\gamma^{-9} - 3\beta\gamma^{-4}\right) I + \left(\alpha\gamma^{-6} + 3\beta\gamma^{-2}\right)\frac{1}{\gamma^2} I - \frac{\beta}{\gamma^4} I\right]$$

$$\sigma_{rr} = \gamma^9 \left[-\alpha\gamma^{-3} + \beta\gamma^{-9} - 3\beta\gamma^{-4} + \alpha\gamma^{-8} + 3\beta\gamma^{-4} - \beta\gamma^{-4} \right]$$

$$= \gamma^9 \left[-\alpha\gamma^{-3} + \beta\gamma^{-9} - \beta\gamma^{-4} + \alpha\gamma^{-8} \right]$$

$$= -\alpha\gamma^6 + \beta - \beta\gamma^5 + \alpha\gamma$$

$$\Rightarrow -p = -(\gamma^5 - 1)(\alpha\gamma + \beta) \ ; \ \alpha > 0, \beta > 0$$

$$\Rightarrow p = (\gamma^5 - 1)(\alpha\gamma + \beta)$$

define a function

$$f(\gamma) = (\gamma^5 - 1)(\alpha\gamma + \beta) - p$$

$$f'(\gamma) = 6\alpha\gamma^5 - \alpha + 5\beta\gamma^4, \ \cancel{= -\alpha}$$

$$f''(\gamma) = 30\alpha\gamma^4 + 20\beta\gamma^3 \ ; \ f''(\gamma) > 0, \ \gamma > 0$$

$\Rightarrow f'(\gamma)$ is increasing for $\gamma > 0$

$\cancel{f'(\gamma)}$ $f'(0) = -\alpha < 0$

This means that, if $f(\gamma) = 0$ for certain value of $\gamma = \gamma' > 0$ that will be the only value for which $f(\gamma) = 0$

Now, $\cancel{f(0) \neq}$ $f(0) = -(\beta + p) < 0$

and $f(\infty) > 0$, since f is continuous, it means $\cancel{f(o}$ $f = 0$, for some $\gamma > 0$.

Now if, $p > 0$

$\cancel{f\theta}$ $f(1) = -p < 0$

and $f(\infty) > 0$

$\Rightarrow f$ is equal to zero for some $1 < \gamma < \infty$.

and since there is only one root > 0, means the only root $> \gamma$.

4) $x_1 = X_1 \cos\tau X_3 - X_2 \sin\tau X_3$ — i

$x_2 = X_1 \sin\tau X_3 + X_2 \cos\tau X_3$ — ii

$x_3 = X_3$ — iii

In radial co-ordinates ($i^2 + ii^2$)

$R \to R$

$\Theta \to \Theta + \frac{\tau L}{L} X_3 = \Theta + \tau X_3$

$z \to Z$

$$F = \text{Grad} \otimes (r e_r + z E_z)$$

$$= \frac{\partial R}{\partial R} e_r \otimes E_r + \frac{R}{R} \frac{\partial e_r}{\partial \Theta} \otimes E_\Theta + \frac{\partial z}{\partial z} E_z \otimes E_z + R \frac{\partial e_r}{\partial z} \otimes E_z$$

$$= e_r \otimes E_r + e_\theta \otimes E_\Theta \frac{\partial \theta}{\partial \Theta} + E_z \otimes E_z$$

$$+ R\, e_\theta \otimes E_z \frac{\partial \theta}{\partial z}$$

$$= e_r \otimes E_r + e_\theta \otimes E_\Theta + E_z \otimes E_z + R\tau e_\theta \otimes E_z$$

$$FF^T = \cancel{e_r \otimes E_r + e} \, (I + R\tau e_\theta \otimes E_z)(I + R\tau E_z \otimes e_\theta$$

$$\Rightarrow B = R\tau e_z \otimes e_\theta + R\tau e_\theta \otimes E_z + I + (R\tau)^2 e_\theta \otimes e_\theta$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+(R\tau)^2 & R\tau \\ 0 & R\tau & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+(r\tau)^2 & r\tau \\ 0 & r\tau & 1 \end{pmatrix}$$

$$\sigma = -pI + (\alpha + \text{tr}\, B) B - \beta B^2$$

Everything, is a function of r (σ is shown in the mathematica output attached)

$FF^T = B$ is given by the following.

```
B = {{1, 0, 0}, {0, 1 + (τ r)^2, τ r}, {0, τ r, 1}}
MatrixForm[B]
```

$\{\{1, 0, 0\}, \{0, 1 + r^2 \tau^2, r \tau\}, \{0, r \tau, 1\}\}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + r^2 \tau^2 & r \tau \\ 0 & r \tau & 1 \end{pmatrix}$$

The stress is given by the following expression (Constitutive law in P.15 on Chadwick).

```
$Assumptions = {α > 0, β > 0, τ > 0}
σ = FullSimplify[-p IdentityMatrix[3] + (α + β Tr[B]) B - β B.B]
MatrixForm[σ]
```

$\{\alpha > 0, \beta > 0, \tau > 0\}$

$$\left\{\left\{\frac{1}{2} (-A^2 + rp^2)\, \alpha\, \tau^2, 0, 0\right\}, \left\{0, \frac{1}{2} (-A^2 + 2 r^2 + rp^2)\, \alpha\, \tau^2, r (\alpha + \beta)\, \tau\right\}, \left\{0, r (\alpha + \beta)\, \tau, -\frac{1}{2} ((A - rp) (A + rp)\, \alpha + 2 r^2 \beta)\, \tau^2\right\}\right\}$$

$$\begin{pmatrix} \frac{1}{2} (-A^2 + rp^2)\, \alpha\, \tau^2 & 0 & 0 \\ 0 & \frac{1}{2} (-A^2 + 2 r^2 + rp^2)\, \alpha\, \tau^2 & r (\alpha + \beta)\, \tau \\ 0 & r (\alpha + \beta)\, \tau & -\frac{1}{2} ((A - rp) (A + rp)\, \alpha + 2 r^2 \beta)\, \tau^2 \end{pmatrix}$$

It can be easily seen that $\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0$, using the divergence equation in z, and θ direction. Hence using $\operatorname{div} \sigma = 0$, $(b=0)$ in r-direction

$$\frac{d\sigma_{rr}}{dr} = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} = r\alpha\tau^2$$

$$\Rightarrow \int_r^A d\sigma_{rr} = \int_r^A \frac{-r\alpha\tau^2 dr}{\cancel{r}}$$

$$\Rightarrow \sigma_{rr}\Big|_r^A = \frac{1}{2}(A-r)(A+r)\alpha\tau^2 \cancel{+\beta\tau^2}$$

$$\Rightarrow \sigma_{rr} = -\frac{1}{2}(A-r)(A+r)\alpha\tau^2 \quad (\sigma_{rr}(A)=0)$$

$$\Rightarrow p = -\sigma_{rr} + (\alpha - \beta + \beta(3 + r^2\tau^2))$$

$$= -\sigma_{rr} + (\alpha + 2\beta + \beta r^2\tau^2)$$

$$\Rightarrow p = \alpha + 2\beta + \frac{1}{2}\tau^2(\alpha(A^2 - r^2) + 2\beta r^2)$$

~~Thus it i~~

The end forces on the faces, are $\sigma \cdot n = (\sigma_{\theta r}, \sigma_{\theta z}, \sigma_{zz})^T$ $(n = 0,0,1)$

($\sigma_{\theta z}$ and σ_{zz})

$\sigma_{\theta z} = r(\alpha+\beta)\tau$ implies,

$\cancel{\sigma_{\theta z} = \frac{1}{2} A^2 + r^2)}$, $t_\theta \; \cancel{\sigma_{\theta z}} = r(\alpha+\beta)\tau . e_\theta$

$\Rightarrow$ shear force, by symmetry $= \int \sigma_{\theta z} 2\pi r dr = 0$

the torque $= \int \sigma_{\theta z} 2\pi r^2 dr$

$$= 2\pi(\alpha+\beta)\tau \int_0^A r^3 dr = \frac{1}{2}(\alpha+\beta)\tau\pi A^4$$

along the z-axis.

Normal force along z-axis is, $\int_0^A t_z \times 2\pi r dr$, $t_z = \sigma_{zz}$

$\cancel{t_z = \int \sigma_{zz} \times 2\pi r dr}$ $\qquad \cancel{= \frac{1}{2}\tau^2 (A^2\alpha - r^2(\alpha+2\beta))}$

$= \int$

$$t_z = -\frac{1}{2}\left((A-r)(A+r)\alpha + 2r^2\beta\right)\tau^2$$

$$\Rightarrow N = \int_0^A 2\pi r\, t_z\, dr$$

$$= -\frac{1}{4}\pi A^4(\alpha + 2\beta)\tau^2$$

These are forces and torques at $x_3 = L$

The torques and forces at $x_3 = 0$ are equal and opposite.

Thus no body forces, or surfaces forces on the cylindrical surface are required.

Also note that a traction is required in ~~x~~ X_3 direction,

The traction goes as τ^2, and is a second-order effect.

5) $$\left.\begin{aligned} x_1 &= \lambda X_1 + a\cos\gamma \\ x_2 &= \lambda X_2 + a\sin\gamma \\ x_3 &= \Lambda X_3 \end{aligned}\right\} \quad \gamma = \frac{kc}{a}\left\{t - \Lambda X_3/c\right\}$$

$$\Rightarrow \begin{aligned} \dot{x}_1 &= -kc\sin\alpha & \Rightarrow \ddot{x}_1 &= a_1 = -\frac{k^2c^2}{a}\cos\alpha \\ \dot{x}_2 &= kc\cos\alpha & \ddot{x}_2 &= a_2 = \frac{k^2c^2}{a}\sin\alpha \\ \dot{x}_3 &= 0 & \ddot{x}_3 &= a_3 = 0 \end{aligned}$$

We obtain F, B, σ etc. using ~~t~~ Mathematica (attached)

$\cdot\ \sigma(x_3) = \sigma(X_3)$

$\operatorname{div}\sigma = \rho a \qquad (b=0)$

$$\Rightarrow \frac{\partial\sigma_{31}}{\partial x_3} = \rho a_1$$

$$\Rightarrow \frac{\partial\sigma_{31}}{\partial X_3}\frac{\partial X_3}{\partial x_3} = -\left(\frac{k\Lambda}{a}\right)\cos\alpha \; k\Lambda^2\left(\phi_1 + (\lambda^2 + (1+k^2)\Lambda^2)\phi_2\right) \times \frac{1}{\Lambda}$$

$$= -\frac{k^2c^2}{a}\cos\alpha * \rho$$

$$\Rightarrow \rho = \frac{a\Lambda^2}{c^2}\left(\phi_1 + (\lambda^2 + (1+k^2)\Lambda^2)\phi_2\right) > 0$$

$$\Rightarrow \phi_1 + (\lambda^2 + (1+k^2)\Lambda^2)\phi_2 > 0 \qquad \left(\begin{aligned}&a>0\\ &\lambda^2, c^2 > 0\end{aligned}\right)$$

The arguments of ϕ_1 and ϕ_2 are I_B, II_B, III_B and are obtained in the mathematica notebook.

The base deformation of the rod is homogenous stretching in X_3 direction of Λ, and stretching ~~shrinking on radius~~ λ along X_1 and X_2 using,

~~$X_1 = R\cos\Theta$~~

~~$X_2 = R\sin\Theta$~~

I Assuming the deformed configuration $\overbrace{\lambda X_1}^{Y_1}, \overbrace{\lambda X_2}^{Y_2}, \overbrace{\Lambda X_3}^{Y_3}$ as base configuration. The displacement field is

$$u_1 = a\cos\gamma \quad , \gamma = \frac{\kappa c}{a}\left[t - \frac{\Lambda X_3}{c}\right] = \frac{\kappa c}{a}\left[t - \frac{Y_3}{c}\right]$$
$$u_2 = a\sin\gamma$$
$$u_3 = 0.$$

at a given section X_3 = constant; the displacement vector will rotate at an angular ~~velocity~~ frequency of $\frac{\kappa c}{a}$

and have an amplitude a. $(\sqrt{u_1^2+u_2^2})$

Now, there is no velocity in X_3-direction.
Now if we ~~fix a point~~,
see how; u_1 varies as a function of Y_3, t

$$u_1 = a\cos\left(\frac{\kappa c}{a}\left(t - \frac{Y_3}{c}\right)\right)$$

This is clearly a wave behavior with velocity c along X_3 direction. And since the direction of propagation is perpendicular to the direction of motion, it is a transverse wave.

```
In[1]:= $Assumptions = {κ > 0, a > 0, c > 0, λ > 0, Λ > 0, ϕ0 > 0, ϕ1 > 0, ϕ2 > 0};
```

The deformation gradient is as follows

```
In[2]:= F = {{λ, 0, κ Λ Sin[γ]}, {0, λ, -κ Λ Cos[γ]}, {0, 0, Λ}};
```

The B matrix is as follows

```
In[3]:= B = F.Transpose[F];
        MatrixForm[B]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} \lambda^2 + \kappa^2 \Lambda^2 \mathrm{Sin}[\gamma]^2 & -\kappa^2 \Lambda^2 \mathrm{Cos}[\gamma]\, \mathrm{Sin}[\gamma] & \kappa \Lambda^2 \mathrm{Sin}[\gamma] \\ -\kappa^2 \Lambda^2 \mathrm{Cos}[\gamma]\, \mathrm{Sin}[\gamma] & \lambda^2 + \kappa^2 \Lambda^2 \mathrm{Cos}[\gamma]^2 & -\kappa \Lambda^2 \mathrm{Cos}[\gamma] \\ \kappa \Lambda^2 \mathrm{Sin}[\gamma] & -\kappa \Lambda^2 \mathrm{Cos}[\gamma] & \Lambda^2 \end{pmatrix}$$

We obtain the three invariants of B

```
In[5]:= IB = FullSimplify[ Tr[B]]
        IIB = FullSimplify[1/2 ((Tr[B])^2 - Tr[B.B])]
        IIIB = FullSimplify[Det[B]]
```

Out[5]= $2 \lambda^2 + (1 + \kappa^2) \Lambda^2$

Out[6]= $\lambda^4 + (2 + \kappa^2) \lambda^2 \Lambda^2$

Out[7]= $\lambda^4 \Lambda^2$

Using the constitute relation in Eq. 34 of Chadwick, we obtain the stress as

```
σ = FullSimplify[ϕ0 IdentityMatrix[3] + ϕ1 B + ϕ2 B.B];
```

Out[9]=

$$\Big\{\Big\{\phi_0 + (\lambda^2 + \kappa^2 \Lambda^2 \mathrm{Sin}[\gamma]^2)\, \phi_1 + (\lambda^4 + \kappa^2 \Lambda^2 (2 \lambda^2 + (1 + \kappa^2) \Lambda^2) \mathrm{Sin}[\gamma]^2)\, \phi_2,$$
$$-\frac{1}{2} \kappa^2 \Lambda^2 \mathrm{Sin}[2 \gamma]\, (\phi_1 + (2 \lambda^2 + (1 + \kappa^2) \Lambda^2)\, \phi_2),\ \kappa \Lambda^2 \mathrm{Sin}[\gamma]\, (\phi_1 + (\lambda^2 + (1 + \kappa^2) \Lambda^2)\, \phi_2)\Big\},$$
$$\Big\{-\frac{1}{2} \kappa^2 \Lambda^2 \mathrm{Sin}[2 \gamma]\, (\phi_1 + (2 \lambda^2 + (1 + \kappa^2) \Lambda^2)\, \phi_2),$$
$$\phi_0 + (\lambda^2 + \kappa^2 \Lambda^2 \mathrm{Cos}[\gamma]^2)\, \phi_1 + (\lambda^4 + \kappa^2 \Lambda^2 (2 \lambda^2 + (1 + \kappa^2) \Lambda^2) \mathrm{Cos}[\gamma]^2)\, \phi_2,$$
$$-\kappa \Lambda^2 \mathrm{Cos}[\gamma]\, (\phi_1 + (\lambda^2 + (1 + \kappa^2) \Lambda^2)\, \phi_2)\Big\},\ \{\kappa \Lambda^2 \mathrm{Sin}[\gamma]\, (\phi_1 + (\lambda^2 + (1 + \kappa^2) \Lambda^2)\, \phi_2),$$
$$-\kappa \Lambda^2 \mathrm{Cos}[\gamma]\, (\phi_1 + (\lambda^2 + (1 + \kappa^2) \Lambda^2)\, \phi_2),\ \phi_0 + \Lambda^2 \phi_1 + (1 + \kappa^2) \Lambda^4 \phi_2\}\Big\}$$