**EN221: Final Examination.**

**Instructions:**

1. This is a take-home exam. You can work on the exam for 24 Hrs. *Please bring your exams to 604 Barus and Holley before Dec 24.*

2. You may refer to the lecture notes, homework solutions, the textbooks (Chadwick and Ogden) and formula sheets that you have prepared based on the class notes. You are not allowed to refer to any other material.

3. There are 4 problems on this exam. Please start each problem on a new sheet of paper.

4. Feel free to use software like Maple, Mathematica, Matlab and so on, but please be sure to attach the input commands and output (files/graphs) from these programs.

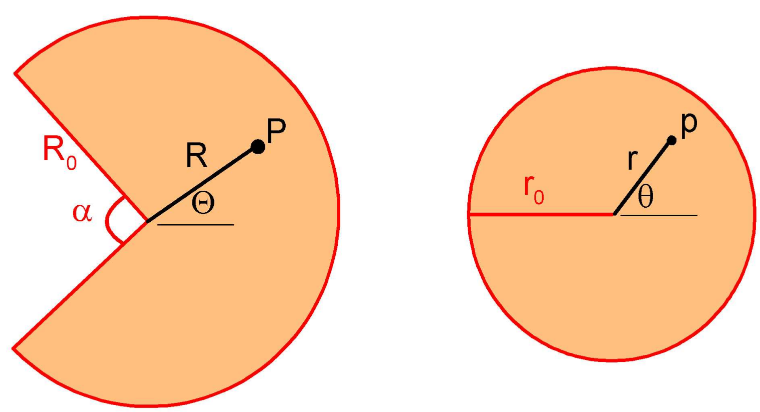
1. (20 Points) A rubber band is held fixed at one end and stretched at the other end at a *constant* speed $U$. The band is aligned with the X-axis, its fixed end is at $X = 0$, and the stretching takes place in the positive $X$-direction. When the stretching starts at $t = 0$ the length of the rubber band is $L_0$ and the velocities of the material points increase in a linear manner from zero at the fixed end to $U$ at $X = L_0$. Note that the velocity of each material point remains *constant* during the deformation process.

   (a) Determine the deformation gradient and the spatial velocity field.

   (b) Suppose that at $t = 0$, an ant jumps on the band and begins to crawl from left to right at a constant speed $u$ *relative to the material points of the deforming band.* Calculate the location $x$ of the ant in the deformed configuration of the rubber band as a function of time.

   (c) Does the ant in (b) ever get to the end of the rubber band. If so, compute the time it takes the ant to completely traverse the rubber band.

2. (30 points) A wedge in an infinitely long cylinder of radius $R_0$ (in the undeformed configuration) is closed by gluing the faces as shown in the figure below. The opening angle of the wedge is given by $\alpha$ (in radians). Let $r$, $\theta$ and $z$ denote the location in the deformed configuration of the material point P with coordinates ($R$,$\Theta$,$Z$) in the undeformed configuration as shown in the figure.




Undeformed          Deformed

(a) If the deformation map is given by

$$r = f(R), \quad \theta = g(\Theta), \quad \text{and} \quad z = Z, \tag{1}$$

obtain the deformation gradient in cylindrical coordinates.

(b) If the cylinder is made out an incompressible material, obtain a condition that the functions $f(R)$ and $g(\Theta)$ has to satisfy. Using this relation, and the geometry of deformation sketched in the figure above, solve for $f(R)$ and $g(\Theta)$.

(c) From your result in (b) what is the radius $r_0$ of the deformed cylinder, if the initial radius is $R_0$ ? Can this result be derived using an independent and perhaps a more elementary approach ?

(d) If the incompressible cylinder satisfies the constitutive relation of a neo-hookean material,

$$W = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \tag{2}$$

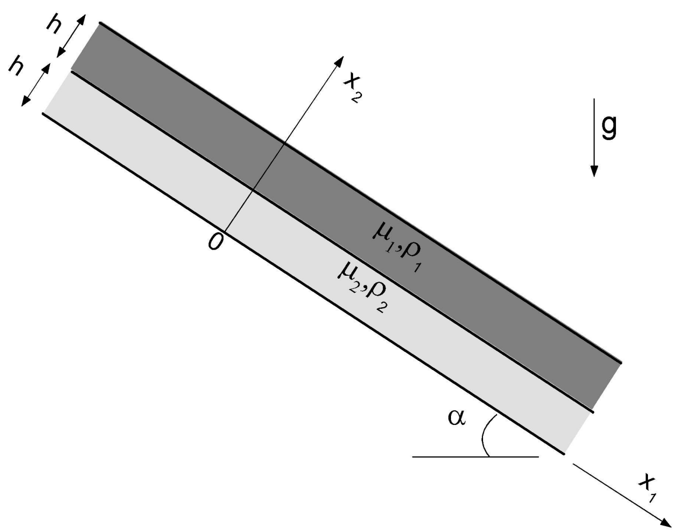
where $\lambda_i$s are the principal stretches, write down an expression for the Cauchy stress. Do not forget to include the pressure term arising from the incompressibility constraint.

(e) Using equilibrium equations in the deformed configuration, obtain, all 9 components of the Cauchy stress for the region $0 \leq r \leq r_0$ in terms of $\mu$, $\alpha$ and $R_0$ or $r_0$.

(f) Is the stress well-behaved or divergent near $r = 0$ ? Give a physical reason for the behavior you observe.

(g) If the glue can only sustain a maximum force per unit length, $F_0$, between the faces of the wedge after the cylinder is deformed, obtain a relation between $F_0$ and the *critical* value of the opening angle, $\alpha_c$, beyond which the wedge cannot be closed.

3. (30 Points) Two incompressible Newtonian fluids which do not mix form two layers, each of uniform thickness $h$, flowing steadily under gravity down a fixed plane inclined to the horizontal at an angle $\alpha$. The densities of the upper and lower fluids are $\rho_1$ and $\rho_2$ respectively and the corresponding viscosities are $\mu_1$ and $\mu_2$.




(a) Assuming that the air above the fluid exerts a uniform pressure, derive the velocity field $v_1(x_2)$ for $0 \leq x_2 \leq 2h$.

(b) Suppose that the fluids are now replaced by ones that are incompressible, but obey the non-linear constitutive law

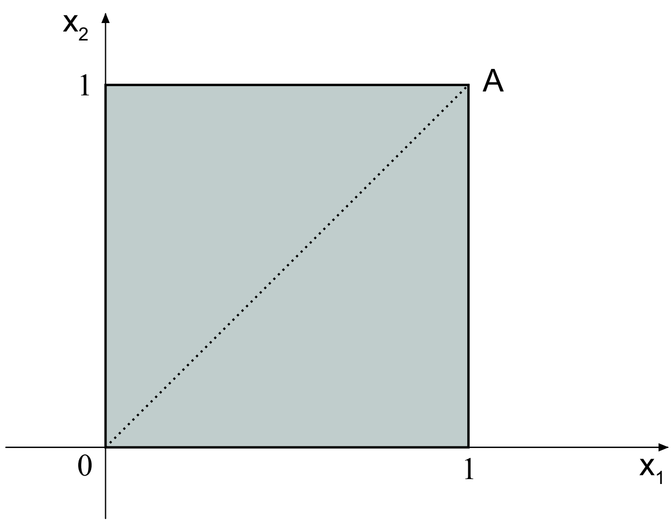
$$\sigma = -p\mathbf{I} + 2\mu_{1(2)}\mathbf{D} + \nu_{1(2)}\mathbf{D}^2, \tag{3}$$

where $\sigma$ is the Cauchy stress, $p$ is the pressure, $\mathbf{D}$ is the stretching tensor and the subscripts refer to the top(bottom) fluid. Is the velocity field for this system same as the field obtained for the Newtonian fluid ? If your answer is yes, explain your reasoning clearly. If your answer is no, obtain the new velocity field.

(c) In the case of the non-linear fluids in (b), is the pressure continuous at the interface between the two fluids *i.e.* at $x_2 = h$ ? If your answer is yes, explain your reasoning clearly. If your answer is no, obtain an expression for the jump in pressure.

4. (20 Points) A thin square sheet of material whose edge length is unity (see the figure below), deforms such that material points are displaced outside the $X_1 - X_2$ plane (in the $X_3$- direction). The displacement field is given by

$$u_1 = 0, \quad u_2 = 0, \quad u_3(X_1, X_2) = X_1 + \frac{X_2^2}{\sqrt{2}}. \tag{4}$$




(a) Calculate the area of the deformed sheet.

(b) Consider the line 0A, whose length in the undeformed configuration is $\sqrt{2}$. This line transforms into a curve on the deformed sheet. What is the total length of this curve in the deformed configuration ?

(c) Suppose that this sheet is made out of a hyperelastic material whose strain energy function is given by

$$W = \frac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 - 2\ln J), \tag{5}$$

where $\lambda_i$ are the principal stretches and $J = \lambda_1\lambda_2\lambda_3$. Calculate the components of the Cauchy stress tensor at the origin due to the deformation.