EN221: HW #1, Due Wednesday, 09/17.

1. Exercise 1.2, Page 47, Chadwick. (Cayley-Hamilton theorem is given by Eq.(58) on page 25.)
2. Exercise 1.3, Page 47, Chadwick. (Hint: Use Eq. 1.1.26 on page 6 of Ogden to do the first part. Make sure that you understand how this equation is obtained.)
3. Exercise 1.4, Page 47, Chadwick.

Note: Pages 47 and 25 of Chadwick and page 6 from Ogden are included in this pdf *where Γ has positive orientation relative to the unit vector field $\boldsymbol{n}$ normal to Λ.*

The reference in the foregoing statement to the orientation (or sense of description) of Γ is illustrated in Figure 2. x, y and z are

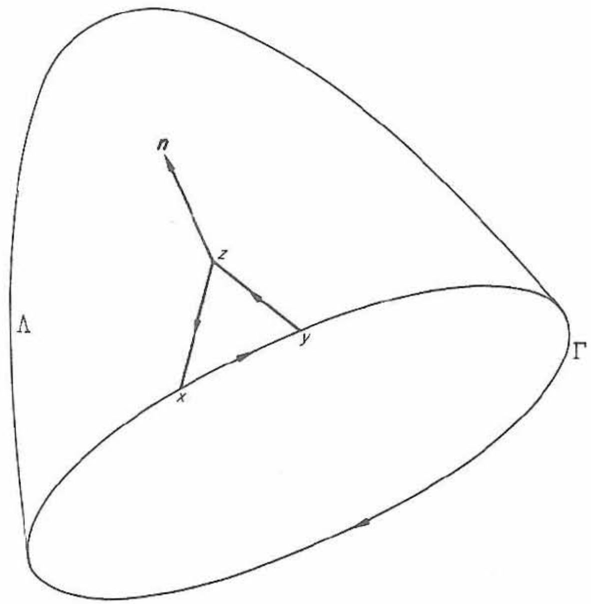

FIGURE 2 Orientation of a simple closed curve Γ relative to a unit normal vector field $\boldsymbol{n}$ on a spanning surface Λ.

nearby points, x and y on Γ and z an interior point of Λ. The indicated sense of description of the circuit xyz, induced by the orientation of Γ, is related to the direction of the unit normal to Λ at z by a right-hand screw rule.[20]

EXERCISES

(The starred exercises contain results which are used in the later chapters.)

[20] In more formal terms, there exists a positive real number ε such that $[\vec{xz}, \vec{yz}, \boldsymbol{n}] > 0$ whenever xz and yz are less than ε.

1*. Show that an arbitrary tensor $\boldsymbol{A}$ can be expressed as the sum of a *spherical tensor* (i.e. a scalar multiple of the identity tensor) and a tensor with zero trace. Prove that this decomposition is unique and that $\boldsymbol{A}'$, the traceless part of $\boldsymbol{A}$, is given by

$$\boldsymbol{A}' = \boldsymbol{A} - \tfrac{1}{3}(\operatorname{tr}\boldsymbol{A})\boldsymbol{I}.$$

[$\boldsymbol{A}'$ is called the *deviator* of $\boldsymbol{A}$.]

2*. Let $\boldsymbol{A}$ be an arbitrary tensor. Show that

$$II_A = \tfrac{1}{2}\{(\operatorname{tr}\boldsymbol{A})^2 - \operatorname{tr}\boldsymbol{A}^2\}.$$

Using the Cayley–Hamilton theorem, deduce that

$$III_A = \tfrac{1}{6}\{(\operatorname{tr}\boldsymbol{A})^3 - 3\operatorname{tr}\boldsymbol{A}\operatorname{tr}\boldsymbol{A}^2 + 2\operatorname{tr}\boldsymbol{A}^3\}.$$

3. Using the result

$$\det\boldsymbol{A}\det\boldsymbol{B} = \det(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}) \quad \forall\, \boldsymbol{A}, \boldsymbol{B} \in \mathsf{L},$$

or otherwise, show that

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}).$$

Hence derive the formulae

(*a*) $\varepsilon_{ijp}\varepsilon_{lmp} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$,

(*b*) $\varepsilon_{ipq}\varepsilon_{lpq} = 2\delta_{il}$.

4. Let $\boldsymbol{A}$ be an arbitrary tensor and $\boldsymbol{A}^*$ its adjugate.

(i) Given that A_{ij} are the components of $\boldsymbol{A}$ relative to an orthonormal basis e, show that the components of $\boldsymbol{A}^*$ are $\tfrac{1}{2}\varepsilon_{ipq}\varepsilon_{jrs}A_{pr}A_{qs}$. Deduce that $\boldsymbol{A}^{\mathrm{T}*} = \boldsymbol{A}^{*\mathrm{T}}$.

(ii) Show that

(*a*) $(\boldsymbol{A}^*)^* = (\det\boldsymbol{A})\boldsymbol{A}$,

(*b*) $\operatorname{tr}\boldsymbol{A}^* = II_A$,

(*c*) $\boldsymbol{A}\{\boldsymbol{a} \wedge (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b})\} = (\boldsymbol{A}^*\boldsymbol{a}) \wedge \boldsymbol{b} \quad \forall\, \boldsymbol{a}, \boldsymbol{b} \in \mathsf{E}$.

5. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be arbitrary tensors, $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$ the adjugates of $\boldsymbol{A}$

The reader is invited to write out the right-hand side of equation (52) as a sum of six terms and to verify that the same answer is obtained on expanding $\varepsilon_{pqr}A_{1p}A_{2q}A_{3r}$. This means that

$$\det A^{\mathrm{T}} = \det A \quad \forall A \in \mathrm{L}. \tag{54}$$

4 PROPER VECTORS AND PROPER NUMBERS OF TENSORS

Let A be an arbitrary tensor. A non-zero vector p is said to be a *proper vector*[8] of A if there exists a scalar (i.e. a *real* number) λ such that

$$Ap = \lambda p, \quad \text{i.e.}\ (A - \lambda I)p = 0; \tag{55}$$

λ is called the *proper number*[8] of A associated with p. Reciprocally, a scalar λ is a proper number of A if there is a non-zero vector p such that (55) holds, and in this situation p is said to be a proper vector of A associated with λ.

Problem 5 (p. 19) implies that λ is a proper number of A if and only if it is a real root of the equation

$$\det(A - \lambda I) = 0.$$

This is known as the *characteristic equation* of A and in view of equation (30) it can also be expressed as

$$[Aa - \lambda a, Ab - \lambda b, Ac - \lambda c] = 0, \tag{56}$$

where a, b, c are arbitrary vectors. On expanding the left-hand side of (56) with the aid of equations (10) and (11), then using the definitions (28) to (30), the arbitrary factor $[a, b, c]$ can be removed and we arrive at the alternative form

$$\lambda^3 - I_A\lambda^2 + II_A\lambda - III_A = 0 \tag{57}$$

of the characteristic equation. Since the principal invariants I_A, II_A, III_A are real, we deduce from equation (57) that A has either three proper numbers or only one.

Problem 10 Let f be a real polynomial, A an arbitrary tensor and λ a proper number of A. Show that $f(\lambda)$ is a proper number of $f(A)$ and that a proper vector of A associated with λ is also a proper vector of $f(A)$ associated with $f(\lambda)$.

[8] The terms *characteristic vector*, *characteristic root* (or *value*) and *eigenvector*, *eigenvalue* are also widely used.

Solution. Let p be a proper vector of A associated with λ. Because of equation (55) the relation

$$A^r p = \lambda^r p \tag{A}$$

holds for $r = 1$. Suppose that it holds for $r = 1, 2, \ldots, n$. Then

$$A^{n+1}p = A(A^n p) = A(\lambda^n p) = \lambda^n Ap = \lambda^{n+1}p,$$

and it may be inferred, by induction, that (A) holds for all positive integers r. Since $f(A)$ is a linear combination of powers of A it follows that $f(\lambda)$ is a proper number of $f(A)$ and p an associated proper vector.

When applied to the characteristic polynomial on the left of equation (57), Problem 10 shows that if A has three proper numbers, the tensor $A^3 - I_A A^2 + II_A A - III_A I$ has three proper numbers each equal to zero. The *Cayley–Hamilton theorem*[9] asserts that, *for arbitrary A*, this tensor is in fact zero;[10] that, in other words, a tensor satisfies its own characteristic equation:

$$A^3 - I_A A^2 + II_A A - III_A I = \mathbf{O} \quad \forall A \in \mathrm{L}. \tag{58}$$

5 SYMMETRIC TENSORS

A symmetric tensor S possesses three proper numbers ($\lambda_1, \lambda_2, \lambda_3$, say) and an orthonormal set of proper vectors, p_1, p_2, p_3, associated respectively with λ_1, λ_2, λ_3.[11] Using successively equations (44), $(46)_1$, (55) and $(42)_1$, we can express S in terms of λ_i and p_i ($i = 1, 2, 3$) as follows:

$$S = SI = S(p_r \otimes p_r) = (Sp_r) \otimes p_r = \sum_{r=1}^{3} \lambda_r(p_r \otimes p_r). \tag{59}$$

[9] See K. Hoffman and R. Kunze, *op. cit.* p. 166.
[10] Problem 9 (p. 22) shows that if u and v are non-zero orthogonal vectors, all the principal invariants of $u \otimes v$ are zero. This furnishes an example of a non-zero tensor possessing three proper numbers all equal to zero.
[11] See K. Hoffman and R. Kunze, *op. cit.* p. 264.

$$\varepsilon_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix} = \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix}$$

Use of this with (1. 5) leads to the representation

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \tag{1.1.26}$$

and on setting $r = k$ and summing over k from 1 to 3 in (1.1.26) we obtain the useful identity

$$\varepsilon_{kij}\varepsilon_{kpq} = \varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}. \tag{1.1.27}$$

With reference to the basis $\{\mathbf{e}_i\}$ the *triple vector product* $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})$ is expanded as

$$u_s\mathbf{e}_s \wedge (\varepsilon_{kpq}v_pw_q\mathbf{e}_k) = \varepsilon_{kpq}\varepsilon_{krs}v_pw_qu_s\mathbf{e}_r$$

by use of (1.1.15), (1.1.17) and (1.1.19). Application of (1.1.27) reduces this to

$$u_qw_qv_p\mathbf{e}_p - u_pv_pw_q\mathbf{e}_q$$

and the identity

$$\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

follows.

1.1.2 Change of basis

We now consider a second (right-handed) orthonormal basis $\{\mathbf{e}'_i\}$ oriented with respect to $\{\mathbf{e}_i\}$ as depicted in Fig. 1.1.

Since $\{\mathbf{e}_i\}$ is a basis, each of $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ is expressible as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We therefore write

$$\mathbf{e}'_i = Q_{ip}\mathbf{e}_p \qquad (i = 1, 2, 3), \tag{1.1.28}$$

and, on taking the dot product of (1.1.28) with $\mathbf{e}_j$, it is seen that the coefficients Q_{ij} are given by

$$Q_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j. \tag{1.1.29}$$

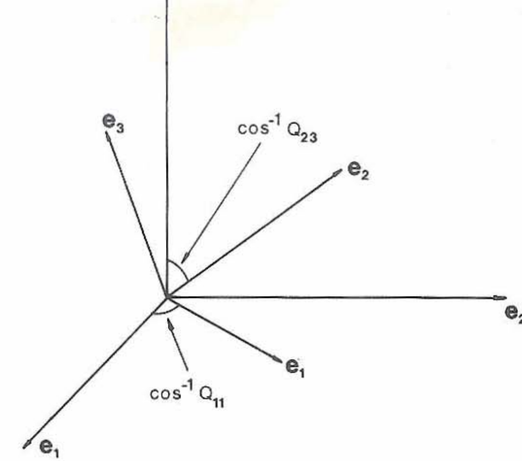

Fig. 1.1 Orientation of the basis vectors $\mathbf{e}'_i$ relative to $\mathbf{e}_i$

The definition (1.1.10) with (1.1.29) shows that the Q_{ij}'s are the direction cosines of the vectors $\mathbf{e}'_i$ relative to the $\mathbf{e}_j$, as indicated in Fig. 1.1.

By orthonormality and (1.1.28), we have

$$\delta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = Q_{ik}\mathbf{e}_k \cdot \mathbf{e}'_j = Q_{ik}Q_{jk}. \tag{1.1.30}$$

It is convenient to represent the collection of coefficients Q_{ij} as a matrix Q, with transpose Q^{T}. Then (1.1.30) shows that Q^{T} is the inverse matrix of Q and so

$$\mathrm{QQ}^{\mathrm{T}} = \mathrm{I} = \mathrm{Q}^{\mathrm{T}}\mathrm{Q}, \tag{1.1.31}$$

where I is the identity matrix, or, in component notation,

$$Q_{ik}Q_{jk} = \delta_{ij} = Q_{ki}Q_{kj}. \tag{1.1.32}$$

A matrix Q satisfying (1.1.31) is said to be an *orthogonal matrix*.

Premultiplication of (1.1.28) by Q_{ij} and use of (1.1.32) leads to the dual connections

$$\mathbf{e}'_i = Q_{ij}\mathbf{e}_j, \qquad \mathbf{e}_j = Q_{ij}\mathbf{e}'_i \tag{1.1.33}$$

between the basis vectors.