# Foundations of Continuum Mechanics

- **Concerned with material bodies (solids and fluids) which can change shape when loaded, and view is taken that bodies are continuous bodies.**
- **Commonly known that matter is made up of discrete particles, and only known continuum is empty space.**
- **Experience has shown that descriptions based on continuum modeling is useful, provided only that <u>variation of field quantities</u> on the <u>scale of deformation mechanism is small</u> in some sense.**

**Goal of continuum mechanics is to solve BVP. The main steps are**

1. **Mathematical preliminaries (Tensor theory)**
2. **Kinematics**
3. **Balance laws and field equations**
4. **Constitutive laws (models)**

**Mathematical structure adopted to ensure that results are <u>coordinate invariant</u> and <u>observer invariant</u> and are consistent with <u>material symmetries</u>.**

# Vector and Tensor Theory

**Most tensors of interest in continuum mechanics are one of the following type:**

1. **Symmetric – have 3 real eigenvalues and orthogonal eigenvectors (eg. Stress)**
2. **Skew-Symmetric – is like a vector, has an associated axial vector (eg. Spin)**
3. **Orthogonal – describes a transformation of basis (eg. Rotation matrix)**

$$\vec{\mathbf{u}} = \sum_{i=1}^{3} \mathbf{u}_i \mathbf{e}_i = \mathbf{u}_i \mathbf{e}_i \qquad \underset{\sim}{\mathbf{T}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{T}_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{T}_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

**When the basis $e_i$ is changed, the components of tensors and vectors transform in a specific way. Certain quantities remain invariant – eg. trace, determinant.**

**Gradient of an $n^{th}$ order tensor is a tensor of order n+1 and divergence of an $n^{th}$ order tensor is a tensor of order n-1.**

$$\nabla \cdot \mathbf{u} = \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_i} \qquad \nabla \otimes \mathbf{u} = \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} \mathbf{e}_i \otimes \mathbf{e}_j \qquad \nabla \cdot \mathbf{T} = \frac{\partial \mathbf{T}_{ij}}{\partial \mathbf{x}_i} \mathbf{e}_j$$

**Integral (divergence) Theorems:**

$$\int_R \nabla \cdot \mathbf{u}\,dv = \int_{\partial R} \mathbf{u} \cdot \mathbf{n}\,da \qquad \int_R \nabla \otimes \mathbf{u}\,dv = \int_{\partial R} \mathbf{u} \otimes \mathbf{n}\,da \qquad \int_R \nabla \cdot \mathbf{T}\,dv = \int_{\partial R} \mathbf{T}^T \mathbf{n}\,da$$

# Kinematics

The tensor that plays the most important role in kinematics is the Deformation Gradient

$$\mathbf{x}(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X}) \qquad \mathbf{F} = \nabla_{\mathbf{X}} \otimes \mathbf{x}(\mathbf{X}) = \mathbf{I} + \nabla_{\mathbf{X}} \otimes \mathbf{u}(\mathbf{X})$$

F can be used to determine

1. Change of lengths, orientations of line segments
2. Change in area, orientations of surfaces
3. Volume changes

F can be written as F = RU (Polar decomposition).
U is a symmetric stretch tensor and R is an orthogonal "rotation" tensor.

Strain measures can be defined using F:

Lagrangean Strain: $$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{T}\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{U}^{2} - \mathbf{I}\right)$$

# Kinematics contd..

**Lagrangean vs Eulerian description:**

1. **In material (Lagrangean ) description, motion (or any other quantity) is written in terms of X and t. Attention is focused on a particle or a region in space as it moves. Usually adopted in Solid mechanics .**

2. **In spatial (Eulerian) description, motion (or any other quantity) is written in terms of x and t. Attention is focused on a point in space, and we study what happens at that point as time changes. Usually adopted in Fluid mechanics.**

**Material derivative:** $$\left.\frac{\partial}{\partial \mathbf{t}}\right|_{\mathbf{X}} = \left.\frac{\partial}{\partial \mathbf{t}}\right|_{\mathbf{x}} + \mathbf{v}\cdot\nabla$$

. **And d/dt are also used to denote material derivative.**

**How does F change with time ?**

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}\ ; \quad \mathbf{L} = \nabla_{\mathbf{x}} \otimes \mathbf{v} \ \ \text{: velocity gradient}$$

# Balance Laws and Field Equations

**1. Conservation of mass:**

$$\dot{\rho} + \rho \nabla_{\mathbf{x}} \cdot \mathbf{v} = 0 \, ; \quad \rho = (\det(\mathbf{F}))\, \rho_{\mathbf{r}}$$

**Spatial** **(1)** **Referential**

**2. Conservation of momentum:**

**Spatial:** $\rho \mathbf{a} = \nabla_{\mathbf{x}} \cdot \sigma + \rho \mathbf{b}\,;$ $\sigma =$ **Cauchy Stress** $\mathbf{b} =$ **body force** **(3)**

**Referential:** $\rho_{\mathbf{r}} \ddot{\mathbf{x}} = \nabla_{\mathbf{X}} \cdot \mathbf{s} + \rho_{\mathbf{r}} \mathbf{b}$ $\mathbf{s} =$ **Nominal Stress** $= \det(\mathbf{F})\mathbf{F}^{-1}\sigma$

**3. Conservation of angular momentum:**

**Spatial:** $\sigma^{T} = \sigma$ ; **(3)** **Referential:** $\mathbf{s}^{T}\mathbf{F}^{T} = \mathbf{F}\mathbf{s}$

**4. Conservation of energy:**

$$\underbrace{\rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)^{\bullet}}_{\text{Kinetic Energy}} + \underbrace{\mathrm{tr}(\sigma \mathbf{L})}_{\text{Stress Power}} = \underbrace{\nabla_{\mathbf{x}} \cdot (\sigma \mathbf{v}) + \rho \mathbf{b} \cdot \mathbf{v}}_{\text{Rate-of-working}}$$

# Constitutive Laws

**# of unknowns: $\rho$ (1) , v (3) , $\sigma$ (9) : 13 unknowns**

**# of field equations: 7**

**→Need constitutive laws (models of material behavior)**

**General form : Stress = material function (deformation)**

**Basic Axiom: A model of material behavior must be invariant under changes in observer. This places restriction on the kind constitutive laws that are admissible.**

**Example: viscous fluid $\sigma$ = -p( $\rho$ ) I + f($\rho$,L)**

$$\mathbf{L} = \mathbf{D} + \mathbf{W}\ ; \quad \mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathsf{T}}\right); \quad \mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^{\mathsf{T}}\right)$$

**Stretch (sym) Spin (anti-sym)**

**Objectivity requires: $\sigma$ = -p( $\rho$ ) I + f($\rho$,D)**

**Example: $\sigma$ = -p( $\rho$ ) I + 2$\mu$D**

# Constitutive Laws for Solids

- **Objectivity – invariance with respect to observer**

$$T(F^*) = QT(F)Q^T = T(QF) \quad \text{for any } Q$$

$$\text{Recall: } \quad F = RU \text{; Choose } Q = R^T$$

$$\Rightarrow T(F) = RT(U)R^T$$

- **Invariance with respect to changing the reference**

$$T(FP) = T(F) \quad \text{for any P (symmetry operation)}$$

$$\text{Recall: } F = VR \quad T(VRP) = T(F)$$

$$\text{If material is isotropic: } P = R^T \quad T(F) = T(V)$$

# Constitutive Laws for Solids Contd..

**Combine conditions from isotropy and material symmetry:**

$$T(V^{*}) = T(QVQ^{T}) = QT(V)Q^{T}$$

$$T(V) = \varphi_0 I + \varphi_1 V + \varphi_2 V^2$$

$\varphi_i$ depend on the principal invariants of $V$

**The principal axes of stress coincide with the Eulerian principal axes**

# Constitutive Equations for Fluids

$$T(L^*) = T(QLQ^T + \dot{Q}Q^T) = QT(L)Q^T$$

$$\text{If } Q = I \text{ and } \dot{Q} = -W, \quad T(L) = T(D)$$

$$\boxed{T(QDQ^T) = QT(D)Q^T}$$

$$\text{Example}: T = -pI + 2\mu D$$