Mathematical Formulas: (bold face denotes vectors)

Particle Dynamics:

Position, velocity, acceleration:

$$\mathbf{R}(t) = \mathbf{R}_0(t) + \mathbf{r}(t), \quad \mathbf{v}(t) = \dot{\mathbf{R}}_0(t) + \dot{\mathbf{r}}(t), \quad \mathbf{a}(t) = \dot{\mathbf{v}}(t)$$

Rotating frames:

$$\dot{\mathbf{b}} = \boldsymbol{\omega} \times \mathbf{b}, \quad \dot{\mathbf{r}} = (\frac{\partial \mathbf{r}}{\partial t})_{oxyz} + \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{v}(t) = \dot{\mathbf{R}}_0 + \mathbf{v}_{rel} + \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a}(t) = \ddot{\mathbf{R}}_0 + \mathbf{a}_{rel} + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Multiple rotating frames:

$$\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}$$

$$\mathbf{v} = \dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2 + \dot{\mathbf{r}}$$

$$\mathbf{v}(t) = \dot{\mathbf{r}}_1 + \mathbf{v}_{2rel} + \boldsymbol{\omega}_1 \times \mathbf{r}_2 + \mathbf{v}_{rel} + \boldsymbol{\omega}_2 \times \mathbf{r}$$

$$\mathbf{a} = \ddot{\mathbf{r}}_1 + \mathbf{a}_{2rel} + 2\boldsymbol{\omega}_1 \times \mathbf{v}_{2rel} + \dot{\boldsymbol{\omega}}_1 \times \mathbf{r}_2 + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_2)$$

$$+\mathbf{a}_{rel} + 2\boldsymbol{\omega}_2 \times \mathbf{v}_{rel} + \dot{\boldsymbol{\omega}}_2 \times \mathbf{r} + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{r})$$

Kinetic energy, potential energy:

$$T = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v}, \quad V(\mathbf{r}) = -\int_{\mathbf{r_0}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}, \quad L = T - V$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} - Q_i^{NC} = 0 \quad i = 1, \ldots, n$$

Holonomic constraints:

$$f_j(q_1, \ldots, q_m, t) = 0 \quad j = 1, \ldots, (m-n)$$

Lagrange Multipliers:

$$S_i = \sum_{k=1}^{N_P} \mathbf{R}^{(k)} \cdot \frac{\partial \mathbf{r}^{(k)}}{\partial q_i} \quad S_i = \sum_{j=1}^{(m-n)} \lambda_j \frac{\partial f_j}{\partial q_i} \quad i = 1, \ldots, m$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} - Q_i^{NC} - \sum_{j=1}^{(m-n)} \lambda_j \frac{\partial f_j}{\partial q_i} = 0, \quad i = 1, \ldots, m,$$

$$\frac{d}{dt}(\frac{\partial L^*}{\partial \dot{q}_i}) - \frac{\partial L^*}{\partial q_i} - Q_i^{NC} = 0 \quad i = 1, \ldots, m \quad \text{where} L^* = L + \sum_{j=1}^{(m-n)} \lambda_j f_j.$$

Rigid Body Dynamics:

Angular momentum, angular velocity, moment of inertia:

$$\dot{\overline{\mathbf{H}}} = \overline{\mathbf{M}} \qquad \dot{\mathbf{H}}_\mathbf{O} = \mathbf{M}_\mathbf{O}$$

$$\overline{\mathbf{H}} = \overline{I}\boldsymbol{\omega} \qquad \dot{\overline{\mathbf{H}}} = \overline{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \overline{I}\boldsymbol{\omega}$$

$$\overline{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \overline{I}\boldsymbol{\omega} = \overline{\mathbf{M}}$$

Euler's Equations relative to principal axes:

$$\overline{I}_{xx}\dot{\omega}_x + (\overline{I}_{zz} - \overline{I}_{yy})\omega_y\omega_z = \overline{M}_x$$

$$\overline{I}_{yy}\dot{\omega}_y + (\overline{I}_{xx} - \overline{I}_{zz})\omega_z\omega_x = \overline{M}_y$$

$$\overline{I}_{zz}\dot{\omega}_z + (\overline{I}_{yy} - \overline{I}_{xx})\omega_x\omega_y = \overline{M}_z$$

Euler angles – angular velocity:

$$\boldsymbol{\omega} = \dot{\psi}\mathbf{K} + \dot{\theta}\mathbf{j}' + \dot{\phi}\mathbf{i}'' = \dot{\psi}\mathbf{e}_\psi + \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\mathbf{e}_\phi = (\dot{\phi} - \dot{\psi}\sin\theta)\mathbf{e}_\phi + \dot{\theta}\mathbf{e}_\theta + \dot{\psi}\cos\theta\hat{\mathbf{e}}^\psi$$

$$\mathbf{e}_\psi \cdot \mathbf{e}_\theta = 0, \quad \mathbf{e}_\theta \cdot \mathbf{e}_\phi = 0, \quad \mathbf{e}_\psi \cdot \mathbf{e}_\phi = -\sin\theta$$

$$\mathbf{e}_\phi \cdot \boldsymbol{\omega} = \dot{\phi} - \dot{\psi}\sin\theta \equiv \Omega$$

Kinetic energy: (axisymmetric body with $\overline{I}_{xx} = I_a, \quad \overline{I}_{yy} = \overline{I}_{zz} = I_t$)

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \overline{\mathbf{I}}\boldsymbol{\omega} = \frac{1}{2}I_a(\dot{\phi} - \dot{\psi}\sin\theta)^2 + \frac{1}{2}I_t(\dot{\theta}^2 + \dot{\psi}^2\cos^2\theta)$$

Angular momenta:

$$p_\psi \equiv \frac{\partial T}{\partial \dot{\psi}} = \mathbf{e}_\psi \cdot \overline{\mathbf{H}} = -I_a\Omega\sin\theta + I_t\dot{\psi}\cos^2\theta$$

$$p_\theta \equiv \frac{\partial T}{\partial \dot{\theta}} = \mathbf{e}_\theta \cdot \overline{\mathbf{H}} = I_t\dot{\theta}$$

$$p_\phi \equiv \frac{\partial T}{\partial \dot{\phi}} = \mathbf{e}_\phi \cdot \overline{\mathbf{H}} = I_a\Omega$$

Waves and vibrations in continuous elastic materials:

Longitudinal waves in bars:

$$\rho\frac{\partial^2 u}{\partial t^2} = E\frac{\partial^2 u}{\partial x^2}$$

or

$$\rho\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = 0$$
$$\frac{1}{E}\frac{\partial \sigma}{\partial t} - \frac{\partial v}{\partial x} = 0$$

Wave speed:

$$\text{elastic bar velocity :} \quad c_0 = \sqrt{\frac{E}{\rho}}$$

Differential relations along characteristics:

$$d\sigma - \rho c_0 dv = 0 \quad \text{along} \quad \frac{dx}{dt} = +c_0$$
$$d\sigma + \rho c_0 dv = 0 \quad \text{along} \quad \frac{dx}{dt} = -c_0$$

Integral relations along characteristics:

$$\sigma - \rho c_0 v = const. \quad \text{along} \quad \frac{dx}{dt} = +c_0$$
$$\sigma + \rho c_0 v = const. \quad \text{along} \quad \frac{dx}{dt} = -c_0$$

Torsional waves in bars:

$$\rho J\frac{\partial \omega}{\partial t} - \frac{\partial T}{\partial z} = 0$$
$$\frac{1}{JG}\frac{\partial T}{\partial t} - \frac{\partial \omega}{\partial z} = 0$$

Wave speed:

$$\text{elastic shear wave velocity :} \quad c_2 = \sqrt{\frac{G}{\rho}}$$

Differential relations along characteristics:

$$dT - \rho c_2 J d\omega = 0 \quad \text{along} \quad \frac{dz}{dt} = +c_2$$
$$dT + \rho c_2 J d\omega = 0 \quad \text{along} \quad \frac{dz}{dt} = -c_2$$

Integral relations along characteristics:

$$T - \rho c_2 J\omega = const. \quad \text{along} \quad \frac{dz}{dt} = +c_2$$
$$T + \rho c_2 J\omega = const. \quad \text{along} \quad \frac{dz}{dt} = -c_2$$

Solution by Separation of Variables:

$$u(x,t) = X(x)T(t); \quad c_0^2 X''(x) + {\omega_n}^2 X(x) = 0; \quad \ddot{T}(t) + {\omega_n}^2 T(t) = 0$$

Wave length, period, frequency, wave number:

$$\lambda_n = \frac{2\pi c_0}{\omega_n}, \quad T_n = \frac{2\pi}{\omega_n}, \quad f_n = \frac{1}{T_n} = \frac{\omega_n}{2\pi}, \quad k_n = \frac{2\pi}{\lambda_n} = \frac{\omega_n}{c_0}$$

Orthogonality of modes $\phi_n(x)$:

$$\int_0^\ell \phi_n(x)\phi_m(x)dx = 0, \quad m \neq n$$

Flexural waves in beams:

$$EI\frac{\partial^4 w}{\partial x^4} - \mu\alpha h^2 \frac{\partial^4 w}{\partial x^2 \partial t^2} + \mu\frac{\partial^2 w}{\partial t^2} = q$$

Harmonic solutions for homogeneous problem $q = 0$:

$$w = Ae^{i(kx-\omega t)}, \quad \text{where} \quad \omega = c_0\left[\frac{\sqrt{\alpha h^2}k^2}{\sqrt{1+\alpha h^2 k^2}}\right]$$

Phase velocity, group velocity:

$$c_p(k) = \frac{\omega}{k}, \quad c_g(k) = \frac{d\omega}{dk} = c_p(k) + k\frac{dc_p}{dk}$$

Vibrations of Multi-Degree-of-Freedom Systems:

Small, free vibrations of conservative systems about equilibrium position:

$$T = \frac{1}{2}\dot{\mathbf{u}}\cdot M\dot{\mathbf{u}}, \quad V = \frac{1}{2}\mathbf{u}\cdot K\mathbf{u}, \quad ; \quad M = M^T, \quad K = K^T, \quad M, K \text{ positive definite}$$

$$L = T - V; \quad M\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{0}$$

Eigenvalues (natural frequencies), eigenvectors (normal modes):

$$\mathbf{u} = \mathbf{U}\,\sin\omega t; \quad \text{non} - \text{trivial solution if } \det[-\omega^2 M + K] = 0$$

$$\omega_1^2,\ \omega_2^2,\ \omega_3^2,\ \ldots,\ \omega_n^2, \quad \text{Real, positive}$$

$$K\mathbf{U}^{(i)} = \omega_i^2 M\mathbf{U}^{(i)}; \quad \mathbf{U}^{(1)},\ \mathbf{U}^{(2)},\ \mathbf{U}^{(3)},\ \ldots,\ \mathbf{U}^{(n)}, \quad \mathbf{U}^{(i)}\cdot M\mathbf{U}^{(j)} = 0 \quad \text{for } i \neq j$$

Forced vibrations:

$$M\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{F}_0 sin\Omega t$$

Steady state solution:

$$\mathbf{u} = \mathbf{U}_0 sin\Omega t; \quad \mathbf{U}_0 = [-\Omega^2 M + K]^{-1}\mathbf{F}_0; \quad \Omega^2 \neq {\omega_i}^2, \quad i = 1, 2, \ldots, n$$

Rayleigh dissipation function:

$$\mathcal{F}(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \equiv \frac{1}{2} \text{ dissipation rate}; \quad Q_i^{NC} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_i}$$

Damped vibrations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = 0, \quad i = 1, 2, \ldots, n$$

$$M\ddot{\mathbf{q}} + C\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{0}; \quad C_{ij} = \frac{\partial^2 \mathcal{F}}{\partial \dot{q}_i \partial \dot{q}_j}$$