Q.1 $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$$\mathbf{v}_{rel} = \dot{\mathbf{r}} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k}, \quad \mathbf{a}_{rel} = \ddot{\mathbf{r}} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}$$

Equation of motion $\mathbf{F} = m\mathbf{a}$

$$\mathbf{a} = \cancelto{0}{\ddot{\mathbf{R}}_0} + \mathbf{a}_{rel} + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} + \cancelto{0}{\dot{\boldsymbol{\omega}} \times \mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \mathbf{i}(\ddot{x} + 2\omega\dot{y} - \omega^2 x) + \mathbf{j}(\ddot{y} - 2\omega\dot{x} - \omega^2 y) + \mathbf{k}(\ddot{z}) \quad \text{—— (1)}$$

$$\mathbf{F} = -T\frac{\mathbf{r}}{|\mathbf{r}|} + mg\,\mathbf{k} = \frac{-T(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})}{\ell} + mg\,\mathbf{k} \quad \text{—— (2)}$$

Equating $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of (1) & (2),

small displacement assumption [$\mathbf{k}$: $-\frac{Tz}{\ell} + mg = m\ddot{z}$, Assuming $z \simeq \ell$, $\ddot{z} = 0 = \dot{z}$ $\Rightarrow$ $T = mg$

$\mathbf{i}$: $\ddot{x} + 2\omega\dot{y} - \omega^2 x = -\frac{gx}{\ell}$ $\Rightarrow$ $\ddot{x} = -2\omega\dot{y} + (\omega^2 - \alpha^2)x$

$\mathbf{j}$: $\ddot{y} - 2\omega\dot{x} - \omega^2 y = -\frac{gy}{\ell}$ $\Rightarrow$ $\ddot{y} = 2\omega\dot{x} + (\omega^2 - \alpha^2)y$

substitute $\dot{x} = u$, $\dot{y} = v$ to get the equations of motion.

Q.2 $T = \frac{1}{2} m\,\mathbf{v}\cdot\mathbf{v}$, $\mathbf{v} = \cancelto{0}{\dot{\mathbf{R}}_0} + \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k} + \omega y\,\mathbf{i} - \omega x\,\mathbf{j}$

$$= \frac{1}{2} m\left[(\dot{x} + \omega y)^2 + (\dot{y} - \omega x)^2 + \dot{z}^2\right]$$

$$\mathbf{F} = -\frac{Tx}{\ell}\mathbf{i} - \frac{Ty}{\ell}\mathbf{j} + \left(mg - \frac{Tz}{\ell}\right)\mathbf{k}, \quad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

$$V = \frac{Tx^2}{2\ell} + \frac{Ty^2}{2\ell} + \frac{T}{2\ell}(z^2 - \ell^2) + mg(\ell - z) \quad \cdots \quad V = -\int_{\text{initial pos.}}^{\text{final pos.}} \mathbf{F}\cdot d\mathbf{r}$$

Q.4 $\mathcal{L} = T - V$

generalized coordinates:

z: $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{z}}\right) - \frac{\partial\mathcal{L}}{\partial z} = 0$ $\Rightarrow$ $m\ddot{z} + \frac{Tz}{\ell} - mg = 0$

For $z \simeq \ell$, $T \triangleq mg$

x, y: $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{L}}{\partial x} = 0$ $\Rightarrow$ $(\ddot{x} + \omega\dot{y}) + (\omega\dot{y} - \omega^2 x) + \frac{gx}{\ell} = 0$

$$\left.\begin{aligned} &\text{i.e.} \quad \ddot{x} = -2\omega\dot{y} + (\omega^2 - \alpha^2)x \\ &\text{and} \quad \ddot{y} = 2\omega\dot{x} + (\omega^2 - \alpha^2)y \end{aligned}\right\} \text{equations same as Q.1}$$

Q.3

$$\mathbf{r} = l\,\mathbf{e}_r$$

$$\dot{\mathbf{r}} = \cancel{\dot{l}}^{\,0}\,\mathbf{e}_r + l\,\dot{\mathbf{e}}_r = l\,\big(\underbrace{\dot\phi\,\mathbf{e}_\phi + \dot\theta\sin\phi\,\mathbf{e}_\theta}_{\text{can be obtained by writing}}\big)$$

$$\mathbf{e}_r = \cos\phi\,\mathbf{k} + \sin\phi\,(\sin\theta\,\mathbf{j} + \cos\theta\,\mathbf{i})$$

$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{k} + \cos\phi\,(\sin\theta\,\mathbf{j} + \cos\theta\,\mathbf{i})$$

$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

$$\mathbf{v} = \cancel{\dot{\mathbf{R}}_0}^{\,0} + \dot{\mathbf{r}} + \boldsymbol{\omega}\times\mathbf{r} = l(\dot\phi\,\mathbf{e}_\phi + \dot\theta\sin\phi\,\mathbf{e}_\theta) + (-\omega l\sin\phi)\,\mathbf{e}_\theta$$

$$= (l\dot\phi)\,\mathbf{e}_\phi + \sin\phi\,(l\dot\theta - \omega l)\,\mathbf{e}_\theta$$

$$T = \frac{1}{2}m\left[(l\dot\phi)^2 + l^2\sin^2\phi\,(\dot\theta - \omega)^2\right]$$

$$\mathbf{F} = -T\,\mathbf{e}_r + mg\,\mathbf{k}, \quad d\mathbf{r} = d\phi\,\mathbf{e}_\phi + d\theta\sin\phi\,\mathbf{e}_\theta$$

$$V = -\int_0^\phi \mathbf{F}\cdot d\mathbf{r} = -\int_0^\phi -mg\sin\phi\,d\phi = mg(1-\cos\phi)$$