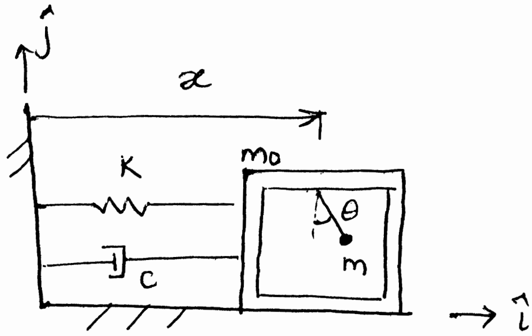
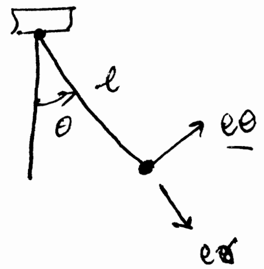
Problem 1

HW6





For mass mo velocity $\underline{v} = \dot{x}\,\hat{i}$

For mass m velocity $\underline{v} = \dot{x}\,\hat{i} + l\dot{\theta}\,\underline{e}_\theta$

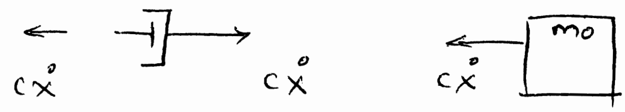
$= (\dot{x} + l\dot{\theta}\cos\theta)\,\hat{i} + l\dot{\theta}\sin\theta\,\hat{j}$

$\Rightarrow$ Kinetic Energy $= T = \frac{1}{2} m_o \dot{x}^2 + \frac{1}{2} m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{x}\cos\theta)$

Potential Energy $= V = \frac{1}{2} k(x - l_o)^2 + mgl(1 - \cos\theta)$

Lagrangian $L = T - V$

Non-conservative force:



$Q_x^{NC}\delta x = -c\dot{x}\,\delta x \Rightarrow Q_x^{NC} = -c\dot{x}, \quad Q_\theta^{NC} = 0$

Then, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_x^{NC}$ &

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_\theta^{NC}$ gives

Equations of motion:

1) $(m + m_o)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + k(x - l_o) + c\dot{x} = 0$

&

2) $l\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0$

Problem 1 Greenwood 6-26.

Kinetic Energy:

$$T = \frac{1}{2}3m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}m\ell^2\dot{\theta}^2 \tag{1.1}$$

Non-holonomic Constraint:

$$\frac{\ell}{2\sqrt{3}}\dot{\theta} - \dot{x}\sin\theta + \dot{y}\cos\theta = 0 \tag{1.2}$$

Lagrange's Equations:

$$3m\ddot{x} = -\lambda\sin\theta \tag{1.3}$$

$$3m\ddot{y} = \lambda\cos\theta \tag{1.4}$$

$$m\ell^2\ddot{\theta} = +\frac{1}{2\sqrt{3}}\ell\lambda \tag{1.5}$$

Eliminate λ:

$$\ddot{x}\cos\theta + \ddot{y}\sin\theta = 0 \tag{1.6}$$

$$\ell\ddot{\theta} + \frac{\sqrt{3}}{2}\ddot{x}\sin\theta - \frac{\sqrt{3}}{2}\ddot{y}\cos\theta = 0 \tag{1.7}$$

Introduce basis $\mathbf{e}_r$, $\mathbf{e}_\theta$:

$$\begin{aligned} \mathbf{e}_r &= \mathbf{i}\cos\theta + \mathbf{j}\sin\theta \\ \mathbf{e}_\theta &= -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta \end{aligned} \tag{1.8}$$

Note that for $\overline{\mathbf{v}}_r = \overline{v}_r\mathbf{e}_r + \overline{v}_\theta\mathbf{e}_\theta$, Eqn. (1.6) corresponds to:

$$\mathbf{e}_r \cdot \mathbf{a} = 0 \Rightarrow \dot{\overline{v}}_r - \dot{\theta}\overline{v}_\theta = 0 \tag{1.9}$$

and Eqn. (1.5) corresponds to:

$$\frac{\ell}{2\sqrt{3}}\dot{\theta} + \overline{v}_\theta = 0. \tag{1.10}$$

Eliminating $\overline{v}_\theta$ from Eqns. one obtains

$$\dot{\overline{v}}_r + \frac{\ell}{2\sqrt{3}}\left(\dot{\theta}\right)^2 = 0 \tag{1.11}$$

whereas differentiation of (1.5) gives

$$\frac{5\ell}{2\sqrt{3}}\ddot{\theta} - \dot{\theta}\overline{v}_r = 0. \tag{1.12}$$

Eliminating $\overline{v}_r$ between (1.11) and (1.12) gives

$$5\dot{\theta}\dddot{\theta} - 5\left(\ddot{\theta}\right)^2 + \left(\dot{\theta}\right)^4 = 0. \tag{1.13}$$

Eqn. (1.13) is a nonlinear ordinary differential equation for $\dot{\theta}(t)$. Because this equation is homogeneous (of degree 4) in the time derivatives, this equation does not determine the time dependence of $\dot{\theta}(t)$. Instead it suggests looking for a solution $\dot{\theta} = \dot{\theta}(\theta)$. Because exponentials can cancel out as a common factor we look for a solution of the form

$$\dot{\theta} = \exp\left[f(\theta)\right] \tag{1.14}$$

for which

$$\begin{aligned} \ddot{\theta} &= \left\{\exp\left[f(\theta)\right]\right\}^2 f'(\theta) \\ \dddot{\theta} &= \left\{\exp\left[f(\theta)\right]\right\}^3 \left[f''(\theta) + 2\left(f'(\theta)\right)^2\right]. \end{aligned} \tag{1.15}$$

Substitution of (1.14) and (1.15) into (1.13) and cancellation of the common exponential term leads to the following differential equation for $f(\theta)$.

$$5f''(\theta) + 5\left[f'(\theta)\right]^2 + 1 = 0 \tag{1.16}$$

Equation (1.16) has the solutions

$$f'(\theta) = \pm\frac{i}{\sqrt{5}}, \; i \equiv \sqrt{-1}. \tag{1.17}$$

Integration of (1.17) gives

$$f(\theta) = \pm i\theta + c \tag{1.18}$$

where c **is a constant. Substitution of (1.18) into (1.14) gives a solution of the form**

$$\dot{\theta} = C\left[\cos\left(\theta/\sqrt{5}\right) \pm i\sin\left(\theta/\sqrt{5}\right)\right] \tag{1.19}$$

where $C \equiv \exp[c]$ **is a constant to be determined from the initial condition for** $\dot{\theta}(0)$, **which, from the initial conditions** $\theta(0) = 0,\ \dot{y}(0) = v_0$, **and the non-holonomic constraint (1.5) is given by**

$$C = -\frac{2\sqrt{3}v_0}{\ell}. \tag{1.20}$$

Substitution of (1.20) into (1.19) and selecting only the real part of (1.19), which is also a solution of (1.13), the solution of (1.13) becomes

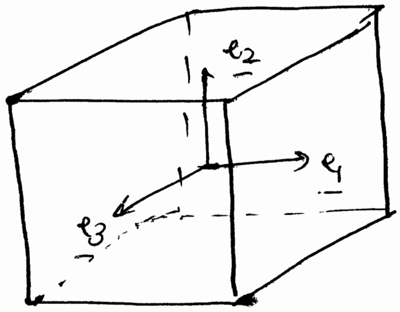
$$\dot{\theta} = -\frac{2\sqrt{3}v_0}{\ell}\cos\left(\theta/\sqrt{5}\right) \tag{1.21}$$

for $\left[\theta/\sqrt{5}\right] \geq -\pi/2;$ **once** θ **becomes equal to** $-\sqrt{5}\pi/2$ **it becomes constant and stays at this value indefinitely. Substitution of (1.21) into (1.4) gives**

$$\lambda = \frac{12\sqrt{3}mv_0^2}{\ell\sqrt{5}}\sin\left(2\theta/\sqrt{5}\right) \tag{1.22}$$

for $\left[\theta/\sqrt{5}\right] \geq -\pi/2;$ **afterwards,** $\lambda = 0,\ \overline{v}_\theta = 0,\ \overline{v}_r = -\sqrt{5}v_0.$

Problem 3



$$\underline{e_1}' = \frac{1}{\sqrt{3}}(\underline{e_1} + \underline{e_2} + \underline{e_3})$$

$$\underline{e_2}' = \frac{1}{\sqrt{6}}(\underline{e_1} + \underline{e_2} - 2\underline{e_3})$$

$$\underline{e_3}' = \frac{1}{\sqrt{2}}(-\underline{e_1} + \underline{e_2})$$

$\bar{I} \Rightarrow \quad I_{xx} = I_{yy} = I_{zz} = \dfrac{ma^2}{6}$ (for cube of side a)

$$[\bar{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{ma^2}{6}$$

$$[\alpha] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$[\bar{I}'] = [\alpha][I][\alpha]^T = [\bar{I}] = \frac{ma^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Since $\bar{I}$ is a diagonal matrix)