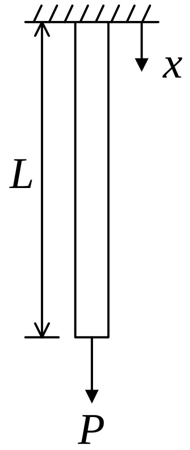
1.2 Introduction to FEM

Finite element method is a very powerful tool for numerical analysis in solid and structural mechanics as well as in many other engineering disciplines. Here we present an introduction to the basic concepts of FEM in the simplest context of a one dimensional elasticity of a bar.

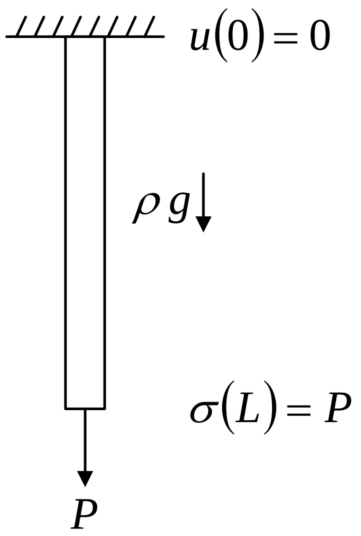
In lecture 1, we have derived the governing equation for 1D elasticity of a bar as,



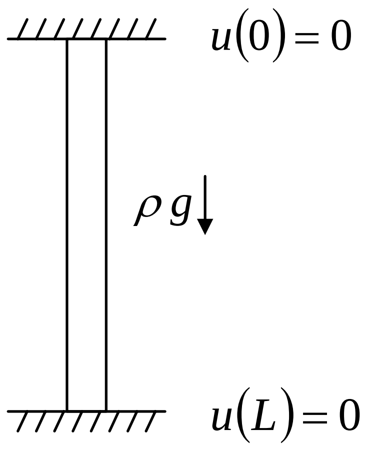
$$\begin{cases} \dfrac{\partial \sigma}{\partial x} + f = \rho \dfrac{\partial^2 u}{\partial t^2} \\ \sigma = E\varepsilon \\ \varepsilon = \dfrac{\partial u}{\partial x} \end{cases}$$

We can consider two simple problems, one with traction BC and one with displacement BC.

P1:



P2:



$$\begin{cases} E u'' + \rho g = 0 \\ u(0) = 0,\ E u'(L) = P \end{cases}$$

$$\begin{cases} E u'' + \rho g = 0 \\ u(0) = u(L) = 0 \end{cases}$$

General solution for $E u'' + \rho g = 0$:

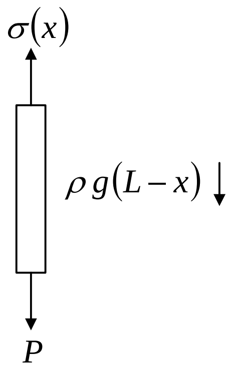
$$u = -\frac{\rho g}{2E} x^2 + C_1 x + \cancelto{0}{C_2}$$

i) Solution for specified boundary conditions of P1:

$$u = -\frac{\rho g}{2E} x^2 + \frac{P}{E} x + \frac{\rho g L}{E} x$$

$$\sigma = E u' = P + \rho g (L - x)$$

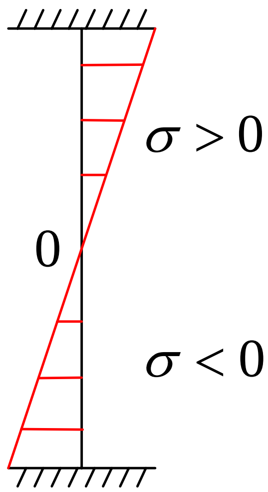
Actually, we can simply get this solution by considering the equilibrium below



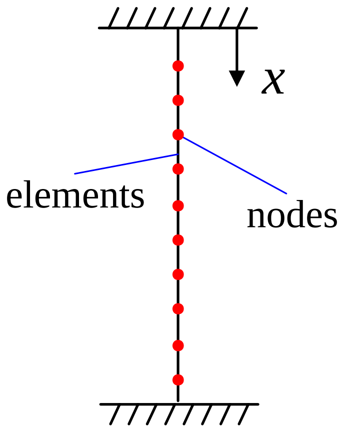
ii) Solution for specified boundary conditions of P2:

$$u = -\frac{\rho g}{2E} x^2 + \frac{\rho g}{2E} L x = \frac{\rho g}{2E} x (L - x)$$

$$\sigma = E u' = \frac{\rho g}{2} (L - 2x)$$



How to use FEM to solve the same problems? (Discretization)



1) Nodal displacement

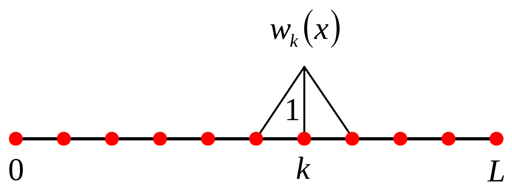
$u_1, u_2, \cdots, u_n$ or $u_j \ (j = 1, 2, \cdots, n)$

$u(x_j) = u_j$

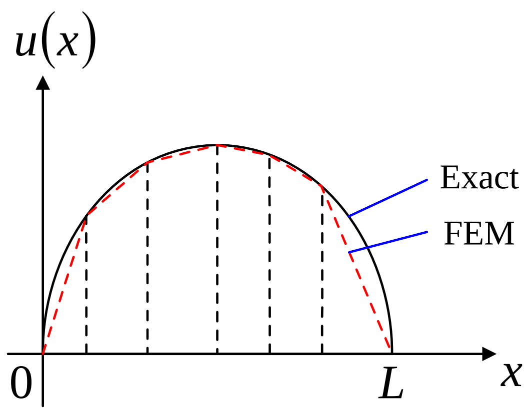
2) Use interpolation to represent $u(x)$ in terms of u_j

$u(x) = \sum_{k=1}^{n} u_k w_k(x)$, where $w_k(x)$ is a function based on element type.

For linear element:



$$w_k(x_j) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$



(More elements lead to more accuracy)

Original problem: to solve $u(x)$ (infinite DOE)

FEM problem: to solve u_j $(j = 1, 2, \cdots, n)$

$Eu'' + f = 0$ (strong form)

$\int_0^L (Eu'' + f) w(x) \mathrm{d}x = 0$ (weak form)

The discrete formula of the weak form is

$\int_0^L (Eu'' + f) w_j(x) \mathrm{d}x = 0$, $(j = 1, 2, \cdots, n)$

$$w_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{x_k - x_{k-1}}, k \in [x_{k-1}, x_k] \\ \dfrac{x_{k+1} - x}{x_{k+1} - x_k}, k \in [x_k, x_{k+1}] \\ \\ 0, \text{otherwise} \end{cases}$$

(n algebraic equations to be solved for n unknowns u_j)

$u(x) = \sum_{k=1}^{n} u_k w_k(x)$, let us calculate the equation in parts.

$$\begin{aligned} \int_0^L E u'' w_j(x) \mathrm{d}x &= E u' w_j(x) \Big|_0^L - \int_0^L E u' w_j'(x) \mathrm{d}x \\ &= -\int_0^L E \sum_{k=1}^{n} w_k'(x) w_j'(x) \mathrm{d}x \, u_k \\ &= -\sum_{k=1}^{n} K_{jk} u_k \end{aligned}$$

where $K_{jk} = \int_0^L E w_k'(x) w_j'(x) \mathrm{d}x$ which is called **Stiffness Matrix**. (It turns out this is still correct even if Young's modulus varies along the length of the bar, i.e. $E = E(x)$. Can u explains why?).

$$\int_0^L f \, w_j(x) \mathrm{d}x = F_j \quad \text{(nodal force)}$$

Therefore, we have

$$\sum_{k=1}^{n} K_{jk} u_k = F_j$$

which can be written in matrix form as

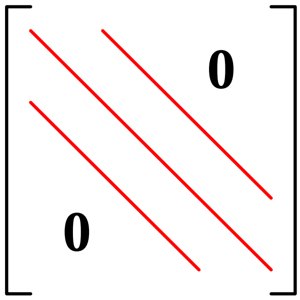
$$\mathbf{KU} = \mathbf{F}$$

where $\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix}$, $\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ \cdots \\ F_n \end{bmatrix}$

Remarks:

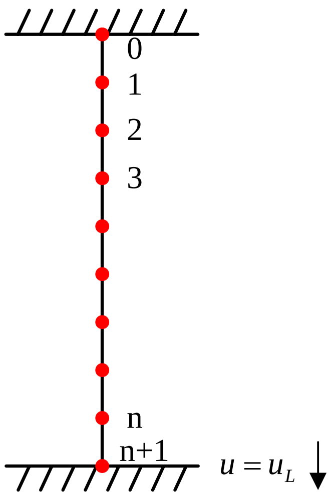
1) $K_{jk} = K_{kj}$ (symmetric matrix)

2) $K_{jk} = 0$, if $|j-k| > 1$

$[\mathbf{K}] =$ 

$\mathbf{K}$ is a sparse matrix. (there is very efficient numerical solver for such a property)

What if $u(L) = u_L \neq 0$?



$$u(x) = \sum_{k=1}^{n} u_k w_k(x) + u_L w_{k+1}(x)$$

$$\int_0^L E u'' w_j(x)\mathrm{d}x = E u' w_j(x)\Big|_0^L - \int_0^L E u' w_j'(x)\mathrm{d}x$$

$$= -\sum_{k=1}^{n} K_{jk} u_k - u_L \int_0^L E\, w_{n+1}'(x) w_j'(x)\mathrm{d}x$$

($j = 1,2,...,n$). If we define the nodal forces as

$$F_j = \int_0^L f\, w_j(x)\mathrm{d}x - u_L \int_0^L E\, w_{n+1}'(x) w_j'(x)\mathrm{d}x$$

The same form $\sum_{k=1}^{n} K_{jk} u_k = F_j$ is obtained.

What if we have traction BC $\sigma = P \,@\, x = L$?

In this case, the nodal displacement at the boundary node $x = L$ is unknown. We need to solve u_j for $j = 1,2,...,n,n+1$.

Consider

$$u(x) = \sum_{k=1}^{n+1} u_k w_k(x)$$

$$\int_0^L E u'' w_j(x)\mathrm{d}x = E u' w_j(x)\Big|_0^L - \int_0^L E u' w_j'(x)\mathrm{d}x$$

$$= P w_j(x_{n+1}) - \sum_{k=1}^{n+1} \int_0^L E w_k'(x) w_j'(x)\mathrm{d}x\, u_k$$

$$= P w_j(x_{n+1}) - \sum_{k=1}^{n+1} K_{jk} u_k$$

Therefore:

$$\sum_{k=1}^{n+1} K_{jk} u_k = P w_j(x_{n+1}) + \int_0^L f\, w_j(x)\mathrm{d}x = F_j$$

What if we are dealing with dynamic problem, i.e. $E u'' + f = \rho \ddot{u}$ ($E\frac{\partial^2 u}{\partial x^2} + f = \rho \frac{\partial^2 u}{\partial t^2}$) ?

$$0 = \int_0^L \left(E u'' + f - \rho \ddot{u}\right) w_j(x)\mathrm{d}x$$

$$\int_0^L E u'' w_j(x)\mathrm{d}x = P(t) w_j(x_{n+1}) - \sum_{k=1}^{n+1} \int_0^L E w_k'(x) w_j'(x)\mathrm{d}x\, u_k(t)$$

The term $\int_0^L f\, w_j(x)\mathrm{d}x$ remains the same as in static case. In the dynamic case, we will have the additional term:

$$-\int_0^L \rho\,\ddot{u}\,w_j(x)\mathrm{d}x = -\sum_{k=1}^{n+1}\int_0^L \rho w_k(x) w_j(x)\mathrm{d}x\,\ddot{u}_k(t)$$
$$= -\sum_{k=1}^{n+1} M_{jk}\,\ddot{u}_k(t)$$

where $M_{jk} = \int_0^L \rho w_k(x) w_j(x)\mathrm{d}x$ is called the **Mass Matrix.** (The density should be kept inside the integral if the density varies along the length of the bar, i.e. $\rho = \rho(x)$.)

Therefore we can write:

$$\sum_{k=1}^{n+1} M_{jk}\,\ddot{u}_k(t) + \sum_{k=1}^{n+1} K_{jk}u_k = P(t)w_j(x_{n+1}) + \int_0^L f\,w_j(x)\mathrm{d}x = F_j(t)$$

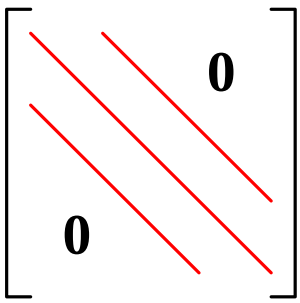
In matrix form

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{F}$$

Remarks:

1) $M_{jk} = M_{kj}$.

2) $M_{jk} = 0$, if $|j-k| > 1$

$[\mathbf{M}] =$ 

Similar to the stiffness matrix, $\mathbf{M}$ is also a symmetric and sparse matrix.