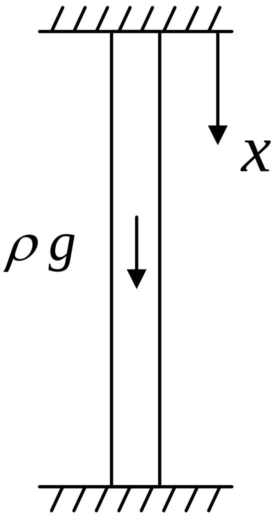
Intro to FEM (continued)

Examples of using FEM to solve a problem and comparison with exact solution:

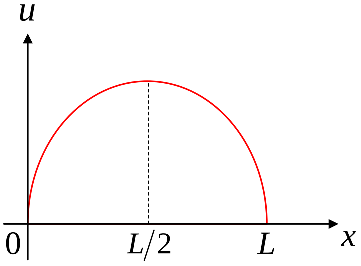
We consider a problem already discussed in the previous class:

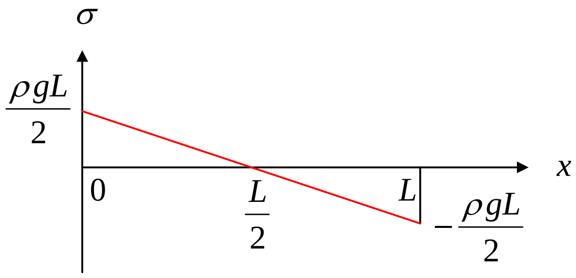


Solution by exact method:

$$u = \frac{\rho g}{2E} x(L - x)$$

$$\sigma = \frac{\rho g}{2}(L - 2x)$$





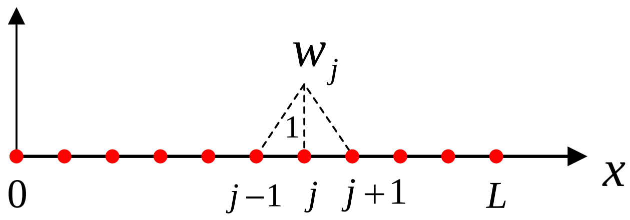
Now we will solve the same problem by FEM:

In FEM, the displacement is discretized as $u(x) = \sum_{k=1}^{n} u_k w_k(x)$ and the governing equation is reduced to algebraic equation:

$$\mathbf{KU} = \mathbf{F} \quad \text{or} \quad \sum_{k=1}^{n} K_{jk} u_k = F_j$$

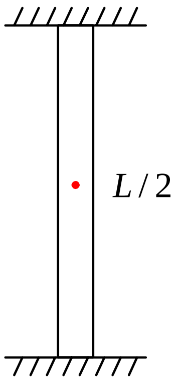
where $K_{jk} = \int_0^L E w_j'(x) w_k'(x) \mathrm{d}x$, $F_j = \int_0^L f\, w_j(x) \mathrm{d}x = \int_0^L \rho g\, w_j(x) \mathrm{d}x$ (see notes of previous lecture).

FEM nodes:



$$w_j(x)=\begin{cases}\dfrac{x-x_{j-1}}{x_j-x_{j-1}}, x\in[x_{j-1},x_j]\\ \dfrac{x_{j+1}-x}{x_{j+1}-x_j}, x\in[x_j,x_{j+1}]\\ 0, \text{otherwise}\end{cases}$$

Case study 1: In the simplest possible case, we choose only one node, i.e. $n=1$. In this case, we have one node and 2 elements.



$$u(x)=u_1 w_1(x)$$

$$w_1(x)=\begin{cases}\dfrac{x}{L/2}, 0<x<L/2\\ \dfrac{L-x}{L/2}, L/2<x<L\end{cases}$$

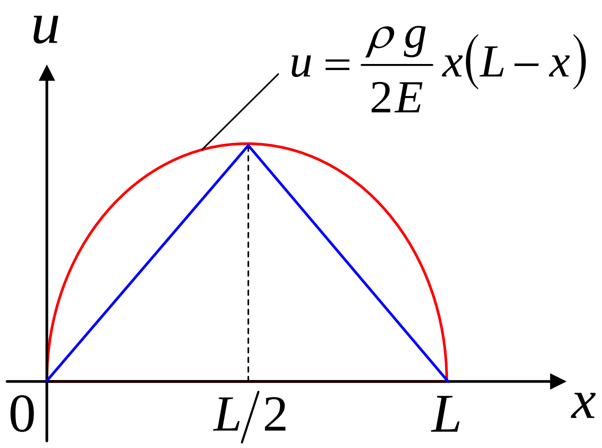
$$w_1'(x)=\begin{cases}\dfrac{2}{L}, 0<x<L/2\\ -\dfrac{2}{L}, L/2<x<L\\ 0, \text{otherwise}\end{cases}$$

$$K_{11}u_1 = F_1$$

$$K_{11} = \int_0^L E w_1'^2(x)\mathrm{d}x = \int_0^L E\left(\frac{2}{L}\right)^2 \mathrm{d}x = 4\frac{E}{L}$$

$$F_1 = \int_0^L \rho g\, w_1(x)\mathrm{d}x = \frac{\rho g L}{2}$$

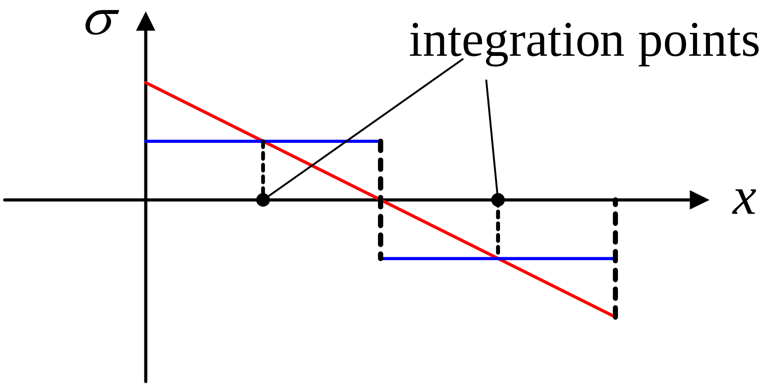
$$u_1 = \frac{F_1}{K_{11}} = \frac{\rho g L^2}{8E}$$



Exact solution: $$u(x = L/2) = \frac{\rho g}{2E}\frac{L}{2}\left(L - \frac{L}{2}\right) = \frac{\rho g L^2}{8E}$$

FEM solution for stress:

$$\sigma(x) = Eu'(x) = Eu_1 w_1'(x) = \begin{cases} 2Eu_1/L, & 0 < x < L/2 \\ -2Eu_1/L, & L/2 < x < L \end{cases}$$
$$= \frac{\rho g L}{4}\begin{cases} 1, & 0 < x < L/2 \\ -1, & L/2 < x < L \end{cases}$$

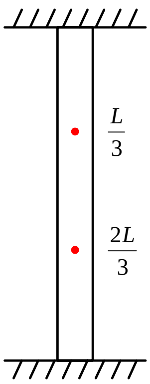


Remarks:

1. $\sigma_{\text{FEM}} = \langle \sigma(x) \rangle_{\text{element}}$ (Stress calculated using FEM corresponding to an average of stress over an element).

2. Stress is not well defined at the nodal points (taking on different values depending on which side of the node). It is better to evaluate stress at integration points.

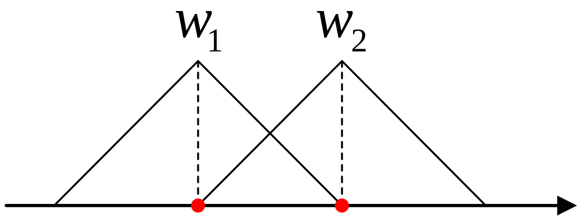
Case study 2: With slightly more sophistication and still within the possibility of doing calculation by hand, we can choose two nodes $n = 2$.



$$u(x) = u_1 w_1(x) + u_2 w_2(x)$$

$$w_1(x) = \begin{cases} \dfrac{x}{L/3}, & 0 < x < L/3 \\ \dfrac{2L/3 - x}{L/3}, & L/3 < x < 2L/3 \\ 0, & 2L/3 < x < L \end{cases}, \quad w_2(x) = \begin{cases} 0, & 2L/3 < x < L \\ \dfrac{x - L/3}{L/3}, & L/3 < x < 2L/3 \\ \dfrac{L - x}{L/3}, & 2L/3 < x < L \end{cases}$$

$$w_1'(x)=\begin{cases}\dfrac{3}{L}, & (0,L/3)\\ -\dfrac{3}{L}, & (L/3,2L/3),\\ 0\,, & (2L/3,L)\end{cases}\qquad w_2'(x)=\begin{cases}0, & (0,L/3)\\ \dfrac{3}{L}, & (L/3,2L/3)\\ -\dfrac{3}{L}, & (2L/3,L)\end{cases}$$



$$\mathbf{K}=\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}$$

$$K_{11}=\int_0^L E w_1'^2(x)\mathrm{d}x=\int_0^{2L/3} E\left(\frac{3}{L}\right)^2\mathrm{d}x=\frac{6E}{L}$$

$$K_{22}=\int_0^L E w_2'^2(x)\mathrm{d}x=\int_{L/3}^{L} E\left(\frac{3}{L}\right)^2\mathrm{d}x=\frac{6E}{L}$$

$$K_{12}=\int_0^L E w_1'(x)w_2'(x)\mathrm{d}x=\int_{L/3}^{2L/3} E\left(-\frac{3}{L}\right)\left(\frac{3}{L}\right)\mathrm{d}x=-\frac{3E}{L}=K_{21}$$

$$\mathbf{K}=\frac{3E}{L}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$$

(Comment: The stiffness matrix is highly banded and sparse, and in some cases can be easily determined. If $n=100$ for the present problem, we can extract the above calculations to determine

$$\mathbf{K} = C_n \underbrace{\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & \mathbf{0} & \\ & -1 & 2 & \ddots & \\ & \mathbf{0} & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}}_{100} \left.\vphantom{\begin{bmatrix} 2 \\ -1 \\ \\ \\ 2 \end{bmatrix}}\right\} 100$$

)

Nodal forces: $F_1 = F_2 = \dfrac{\rho g L}{3}$

$\mathbf{KU} = \mathbf{F}$

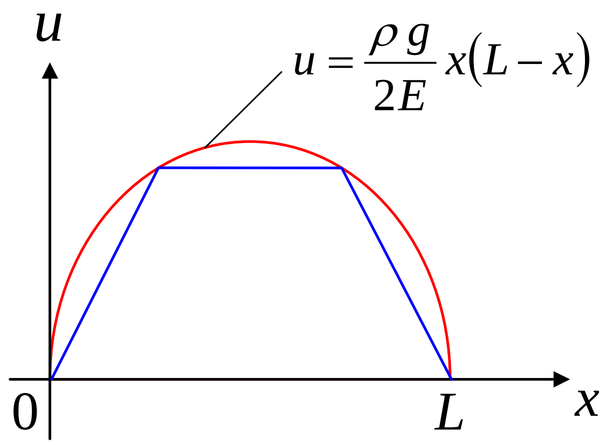
$$\frac{3E}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{\rho g L}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{\rho g L^2}{9E}\cdot\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\rho g L^2}{9E}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u_1 = u_2 = \frac{\rho g L^2}{9E}$$

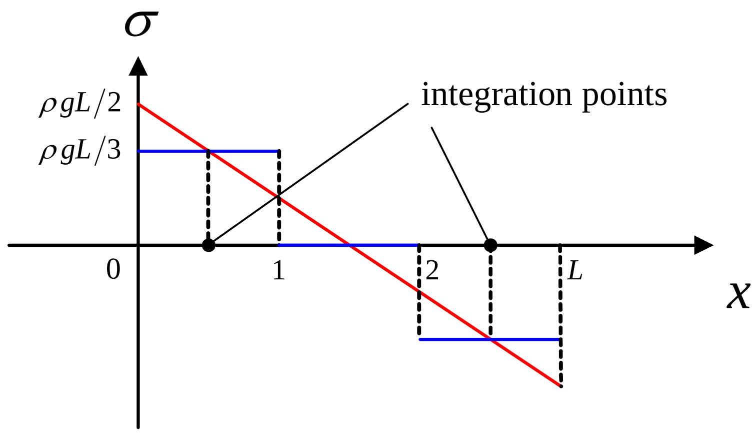
Exact solution: $u(x = L/3) = \dfrac{\rho g}{2E}\dfrac{L}{3}\left(L - \dfrac{L}{3}\right) = \dfrac{\rho g L^2}{9E}$

$$u(x = 2L/3) = \frac{\rho g}{2E}\frac{2L}{3}\left(L - \frac{2L}{3}\right) = \frac{\rho g L^2}{9E}$$

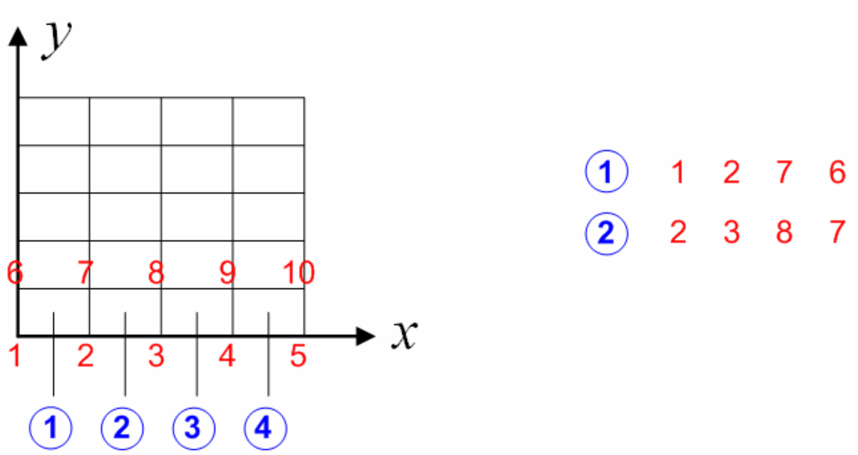


FEM solution for stress:

$$\sigma(x) = Eu'(x) = Eu_1 w_1'(x) + Eu_2 w_2'(x)$$



Brief overview of FEM in 2D and 3D (read course notes Section 1.2 on the web):



Node number (integer);

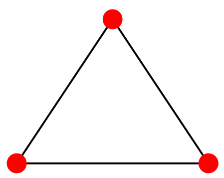
Nodal displacement (vector);
(can also include other physical quantities e.g. temperature, pressure, etc.)

Elements: associated with a number of nodes;

Element connectivity;

Type of elements:

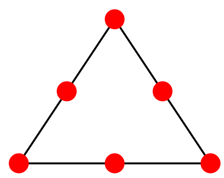
Linear elements:



3 nodal triangle (linear)

$$w(x, y) = a_1 + a_2 x + a_3 y$$

Quadratic elements:



6 nodal triangle

$$w(x, y) = a_1 + a_2 x + a_{3_4} y + a_4 xy + a_5 x^2 + a_6 y^2$$