Continuing on concepts of stress in a continuum:

Traction on an arbitrary plane with normal vector $\vec{n}$

$$t_j = \sigma_{ij} n_i$$

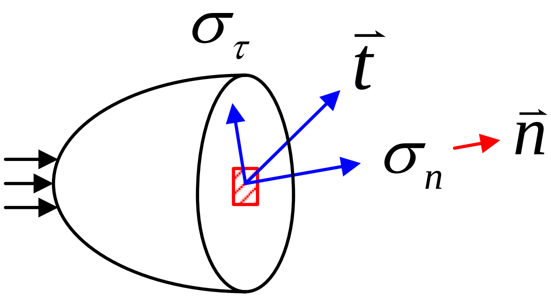
$$\vec{t} = \underline{\sigma}^T \vec{n}$$

This equation indicates that the normal stress on a plane with normal vector $\vec{n}$ is

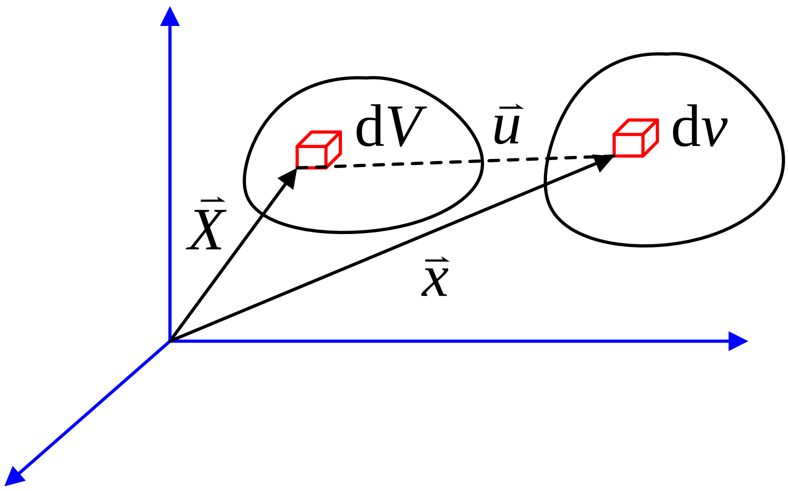
$$\sigma_n = \vec{t} \cdot \vec{n} = \vec{n} \cdot \underline{\sigma}^T \vec{n}$$

Accordingly, the shear stress on a plane with normal $\vec{n}$ is

$$\sigma_\tau \hat{\vec{t}} = \vec{t} - \sigma_n \vec{n} = \underline{\sigma}^T \vec{n} - \left(\vec{n} \cdot \underline{\sigma}^T \vec{n} \right) \vec{n}$$



Force equilibrium (i.e. application of Newton's law $\vec{F} = m\vec{a}$ to a continuum; also called balance of linear momentum)



$\vec{u}$ —— displacement vector

$\dfrac{\partial \vec{u}}{\partial t} = \dot{\vec{u}}$ —— velocity vector

$\dfrac{\partial^2 \vec{u}}{\partial t^2} = \ddot{\vec{u}}$ —— acceleration vector

$\int_V \rho \ddot{\vec{u}} \,\mathrm{d}V$ —— total inertia force on volume V ($\mathrm{d}m = \rho \,\mathrm{d}V$, $a = \ddot{\vec{u}}$)

Consider all the forces acting on the volume:

$\int_S \vec{t} \,\mathrm{d}S$ —— total surface force on V

$\int_V \vec{f} \,\mathrm{d}V$ —— total body force on V

According to Newton's law,

$$\int_S \vec{t} \,\mathrm{d}S + \int_V \vec{f} \,\mathrm{d}V = \int_V \rho \ddot{\vec{u}} \,\mathrm{d}V$$

In index notation:

$$\int_S t_j \,\mathrm{d}S + \int_V f_j \,\mathrm{d}V = \int_V \rho \ddot{u}_j \,\mathrm{d}V$$

The first term is

$$\int_S t_j \,\mathrm{d}S = \int_S \sigma_{ij} n_i \,\mathrm{d}S = \int_V \sigma_{ij,i} \,\mathrm{d}V \quad \text{(Divergence theorem)}$$

Thus

$$\int_V \left(\sigma_{ij,i} + f_j - \rho \ddot{u}_j \right) \mathrm{d}V = 0$$

Since this is true for any volume V (e.g., we can shrink V to a point), the integrand must vanish every point in the continuum,

$$\sigma_{ij,i} + f_j = \rho \ddot{u}_j$$

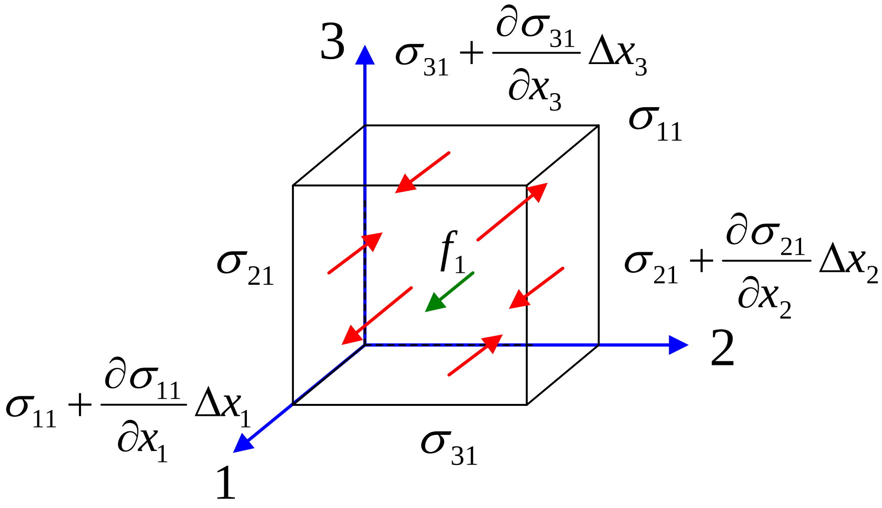
i.e.

$$\frac{\partial \sigma_{ij}}{\partial x_i} + f_j = \rho \ddot{u}_j$$

Compare this with the equation we derived earlier for the 1D case: $\dfrac{\partial \sigma}{\partial x} + f = \rho \ddot{u}$

Classical derivation of equilibrium equations for a continuum:

Consider all the forces in the x_1 acting on a small block of material shown below:



Summing up all the forces in x_1 direction and applying Newton's law,

$$\left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1}\Delta x_1 - \sigma_{11}\right)\Delta x_2 \Delta x_3 + \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2}\Delta x_2 - \sigma_{21}\right)\Delta x_1 \Delta x_3 +$$
$$\left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3}\Delta x_3 - \sigma_{31}\right)\Delta x_1 \Delta x_2 + f_1 \Delta x_1 \Delta x_2 \Delta x_3 = \rho \Delta x_1 \Delta x_2 \Delta x_3 \ddot{u}_1 .$$

This leads to

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = \rho \ddot{u}_1$$

Similarly,

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = \rho \ddot{u}_2$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = \rho \ddot{u}_3$$

i.e.

$$\frac{\partial \sigma_{ij}}{\partial x_i} + f_j = \rho \ddot{u}_j$$

Moment equilibrium (Balance of angular momentum)

Net moment of all forces should vanish at equilibrium

$$\int_S \vec{x} \times \vec{t}\ \mathrm{d}S + \int_V \vec{x} \times \vec{f}\ \mathrm{d}V = \int_V \vec{x} \times \rho \ddot{\vec{u}}\ \mathrm{d}V$$

In index notation:

$$\int_S \varepsilon_{ijk} x_i t_j \vec{e}_k\ \mathrm{d}S + \int_V \varepsilon_{ijk} x_i f_j \vec{e}_k\ \mathrm{d}V = \int_V \varepsilon_{ijk} x_i \rho \ddot{u}_j \vec{e}_k\ \mathrm{d}V$$

Canceling $\vec{e}_k$ at both sides,

$$\int_S \varepsilon_{ijk} x_i t_j\ \mathrm{d}S + \int_V \varepsilon_{ijk} x_i f_j\ \mathrm{d}V = \int_V \varepsilon_{ijk} x_i \rho \ddot{u}_j\ \mathrm{d}V$$

The first term is

$$\begin{aligned}\int_S \varepsilon_{ijk} x_i \sigma_{pj} n_p \mathrm{d}S &= \int_V \left(\varepsilon_{ijk} x_i \sigma_{pj}\right)_{,p} \mathrm{d}V \\ &= \int_V \varepsilon_{ijk} \left(\delta_{ip} \sigma_{pj} + x_i \sigma_{pj,p}\right) \mathrm{d}V \\ &= \int_V \varepsilon_{ijk} \sigma_{ij} + \varepsilon_{ijk} x_i \sigma_{pj,p}\ \mathrm{d}V\end{aligned}$$

Thus

$$\int_V \left[\varepsilon_{ijk} \sigma_{ij} + \varepsilon_{ijk} x_i \left(\sigma_{pj,p} + f_j - \rho \ddot{u}_j\right)\right] \mathrm{d}V = 0$$

$$\Rightarrow \varepsilon_{ijk} \sigma_{ij} = 0$$

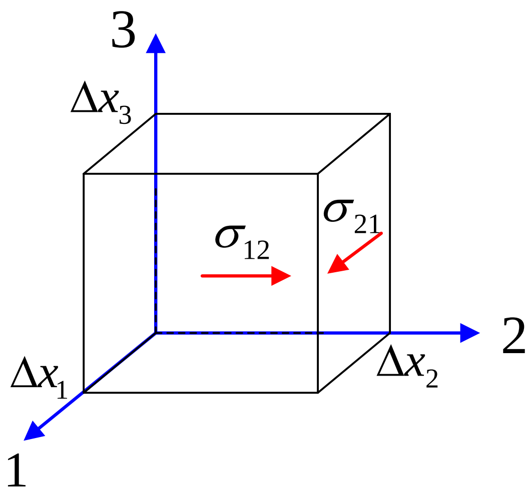
$k = 1$, $\varepsilon_{ij1} \sigma_{ij} = \sigma_{23} - \sigma_{32} = 0 \Rightarrow \sigma_{23} = \sigma_{32}$

$k = 2$, $\varepsilon_{ij2} \sigma_{ij} = \sigma_{31} - \sigma_{13} = 0 \Rightarrow \sigma_{31} = \sigma_{13}$

$k = 3$, $\varepsilon_{ij3} \sigma_{ij} = \sigma_{12} - \sigma_{21} = 0 \Rightarrow \sigma_{12} = \sigma_{21}$

Therefore, balance of angular momentum states that stress tensor is symmetric, i.e.

$$\sigma_{ij} = \sigma_{ji}$$



A simpler demonstration of symmetry of stress tensor:

$$(\sigma_{12}\Delta x_2 \Delta x_3)\cdot \Delta x_1 = (\sigma_{21}\Delta x_1 \Delta x_3)\cdot \Delta x_2$$
$$\Rightarrow \sigma_{12} = \sigma_{21}$$

$t_i = \sigma_{ji} n_j$

$\vec{t} = \underline{\sigma}^T \vec{n} = \underline{\sigma}\,\vec{n}$ ($\underline{\sigma}$ is symmetric)

Eigenvalues of a matrix

$$\underline{T}\,\vec{\xi} = \lambda\,\vec{\xi}$$

This has a very clear meaning for stress tensors. The eigenvalues of a stress tensor are called principal stresses, corresponding to the normal stresses on planes with no shear stresses, i.e.

$$\vec{t} = \underline{\sigma}\,\vec{n} = \sigma\,\vec{n}$$

There in general exist 3 principal stresses and 3 mutually orthogonal principal directions.

Writing $\underline{\sigma}\,\vec{n} = \sigma\,\vec{n}$ in matrix form:

$$\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

For nontrivial solution of $\vec{n}$,

$$\begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma \end{vmatrix} = 0 \qquad (*)$$

Let the solution to equation (*) be $\sigma_1 \neq \sigma_2 \neq \sigma_3$,

$\underline{\sigma}\,\vec{n}^{(1)} = \sigma_1\,\vec{n}^{(1)}$ (1)

$\underline{\sigma}\,\vec{n}^{(2)} = \sigma_2\,\vec{n}^{(2)}$ (2)

$\vec{n}^{(2)} \cdot (1) - \vec{n}^{(1)} \cdot (2)$ leads to

$$(\sigma_1 - \sigma_2)\vec{n}^{(1)} \cdot \vec{n}^{(2)} = 0$$

which shows that, if $\sigma_1 \neq \sigma_2$, we must have $\vec{n}^{(1)} \cdot \vec{n}^{(2)} = 0$. Therefore, $\vec{n}^{(1)}$, $\vec{n}^{(2)}$ are orthogonal vectors, $\vec{n}^{(1)} \perp \vec{n}^{(2)}$. Similarly, we can show $\vec{n}^{(1)} \perp \vec{n}^{(3)}$ and $\vec{n}^{(2)} \perp \vec{n}^{(3)}$.