Mechanical Behavior of Solids

Linear Elastic solids

$$\sigma = E\varepsilon \quad \text{(1D)} \quad \Rightarrow \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{or} \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl} \quad \text{(3D)}$$

where $\underline{C}$ is sometimes called the stiffness tensor and $\underline{S}$ is sometimes called the compliance tensor. Both of them are 4th order elastic moduli tensors.

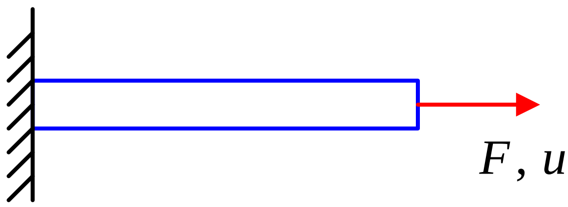
Symmetry of elastic moduli tensors:

$C_{ijkl} = C_{jikl}$, $C_{ijkl} = C_{ijlk}$ (minor symmetry)

The minor symmetries reduce the independent elastic constants from 81 to 36.

There is also a major symmetr in elastic moduli $C_{ijkl} = C_{klij}$, which reduces the number of independent elastic constants from 36 to 21.

We use the concept of energy & work to demonstrate the major symmetry of $\underline{C}$ and $\underline{S}$:



Assume an increment of displacement at the bar end,

$$u \to u + \delta u$$

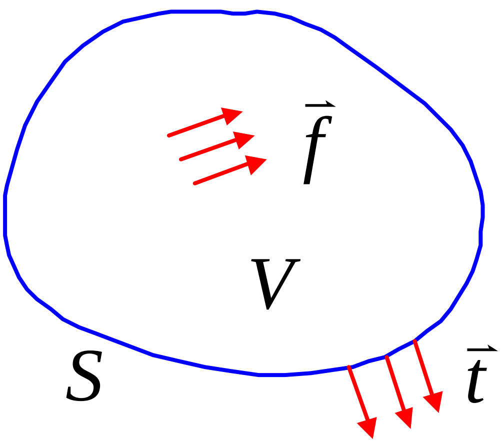
The work done by the applied load should equal to the stored energy in the material,

$$\delta W = F\delta u = \sigma A \delta(l\varepsilon) = Al\sigma\delta\varepsilon = V\sigma\delta\varepsilon$$

$\delta w = \dfrac{\delta W}{V} = \sigma\delta\varepsilon$ should be the stored elastic energy per unit volume, which is also called the strain energy density.

$$w = w(\varepsilon), \quad \sigma = \frac{\partial w}{\partial \varepsilon}, \quad w(\varepsilon) = \int_0^{\varepsilon} \sigma \, d\varepsilon$$

Generalize to 3D



$$\begin{aligned}
\delta w &= \int_S \vec{t} \cdot \delta\vec{u}\, \mathrm{d}S + \int_V \vec{f} \cdot \delta\vec{u}\, \mathrm{d}V \\
&= \int_S t_i \delta u_i\, \mathrm{d}S + \int_V f_i \delta u_i\, \mathrm{d}V \\
&= \int_S \sigma_{ij} n_j \delta u_i\, \mathrm{d}S + \int_V f_i \delta u_i\, \mathrm{d}V \\
&= \int_V \left(\sigma_{ij} \delta u_i\right)_{,j} \mathrm{d}V + \int_V f_i \delta u_i\, \mathrm{d}V \\
&= \int_V \left(\sigma_{ij,j} + f_i\right) \delta u_i \mathrm{d}V + \int_V \sigma_{ij} \delta u_{i,j}\, \mathrm{d}V \\
&= \int_V \sigma_{ij} \delta \varepsilon_{ij}\, \mathrm{d}V = \int_V \delta w\, \mathrm{d}V
\end{aligned}$$

$\delta w = \sigma_{ij} \delta \varepsilon_{ij}$ is the strain energy density and

$w = \int \sigma_{ij} \delta \varepsilon_{ij}$ must be integrable.

Knowing $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$, we have

$$w = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$$

$$C_{ijkl} = \frac{\partial^2 w}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = C_{klij}$$

Elastic moduli for isotropic materials:

Isotropic tensor: $\underline{I}\,\vec{a} = \vec{a}$ ($\delta_{ij} a_j = a_i$)

The 4th order isotropic tensors can be generally written as

$$C_{ijkl} = c_1 \delta_{ij} \delta_{kl} + c_2 \delta_{ik} \delta_{jl} + c_3 \delta_{il} \delta_{jk}$$

Imposing the symmetry conditions indicate that the elastic moduli can be expressed in terms of just two constants

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

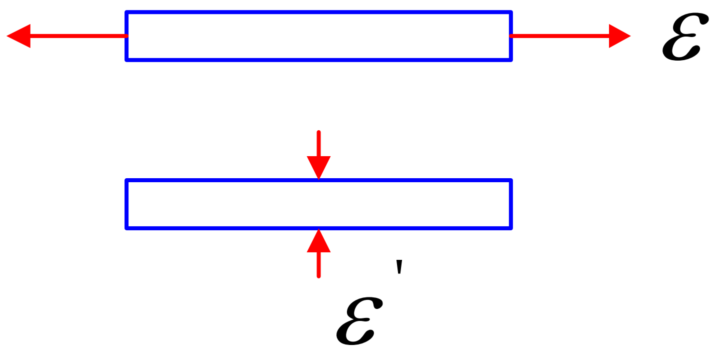
where λ and μ are called Lamé constants.

Similarly, the S_{ijkl} can also be expressed in terms of just two constants as

$$S_{ijkl} = d_1\delta_{ij}\delta_{kl} + d_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

The constants $\lambda, \mu,$ d_1 and d_2 are to be determined experiments.

Since there are only 2 independent elastic constants, it suffices to conduct experiments in 1D:



$$E = \frac{\sigma}{\varepsilon} \quad \text{(Young's modulus)}$$

$$\nu = -\frac{\varepsilon'}{\varepsilon} \quad \text{(Poisson's ratio)}$$

:
We can specify the general stress-strain relation

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} = \left(d_1\delta_{ij}\delta_{kl} + d_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right)\sigma_{kl} = d_1\sigma_{kk}\delta_{ij} + 2d_2\sigma_{ij}$$

to 1D loading $\sigma_{11} \neq 0, \sigma_{22} = \sigma_{33} = 0$. In this case,

$$\varepsilon_{11} = d_1\sigma_{11} + 2d_2\sigma_{11} = \left(d_1 + 2d_2\right)\sigma_{11} = \frac{\sigma_{11}}{E}$$

$$\varepsilon_{22} = d_1\sigma_{11} = -\nu\varepsilon_{11} = -\nu\frac{\sigma_{11}}{E}$$

Therefore,

$$d_1 = -\frac{\nu}{E}$$
$$d_2 = \frac{1+\nu}{2E}$$

and

$$S_{ijkl} = -\frac{\nu}{E}\delta_{ij}\delta_{kl} + \frac{1+\nu}{2E}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

We thus obtain the so-called generalized Hooke's law:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

This equation can be inverted as follows.

$$\varepsilon_{qq} = \frac{1+\nu}{E}\sigma_{qq} - \frac{3\nu}{E}\sigma_{kk} = \frac{1-2\nu}{E}\sigma_{qq}$$

The percentage change of volume,

$$\frac{\Delta V}{V} = \varepsilon_{kk} = \frac{1-2\nu}{E}\sigma_{kk} = \frac{3(1-2\nu)}{E}p$$

Defines the so-called bulk modulu $K = \frac{p}{\Delta V/V} = \frac{E}{3(1-2\nu)}$. (This modulus can also be calculated using ab initio quantum mechanics techniques.)

Invert the generalized Hooke's law:

$$\frac{1+\nu}{E}\sigma_{ij} = \varepsilon_{ij} + \frac{\nu}{E}\sigma_{kk}\delta_{ij} = \varepsilon_{ij} + \frac{\nu}{E}\frac{E}{1-2\nu}\varepsilon_{kk}\delta_{ij}$$
$$\Rightarrow \sigma_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$$

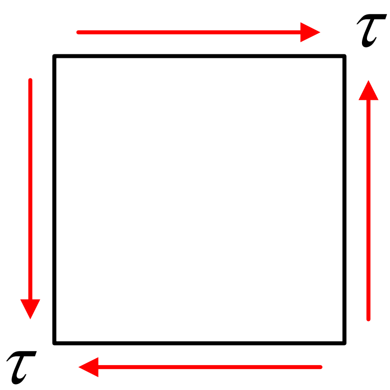
Comparing this to the Lamé form of Hooke's law

$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = \left(\lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right)\varepsilon_{kl} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$, we identify

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

For shear deformation:



$$\gamma = \frac{\tau}{G} \Rightarrow G = \frac{\tau}{\gamma}$$

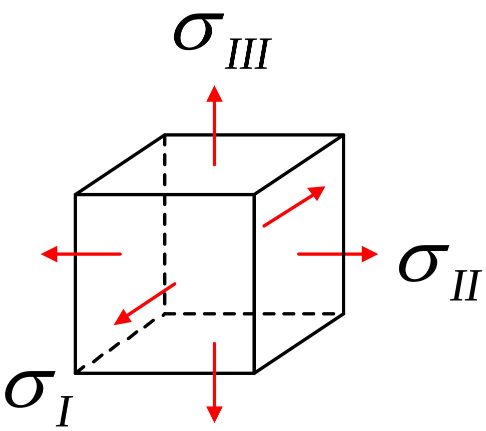
Let $i = 1$, $j = 2$ in $\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$,

$$\sigma_{12} = \tau = 2\mu\varepsilon_{12} = \mu\gamma$$

Therefore, the second Lamé constant is just the shear modulus:

$$G = \mu = \frac{E}{2(1+\nu)}$$

Alternative derivation of the generalized Hooke's law



Consider a linear superposition of strain in the principal stress directions

$$\varepsilon_I = \frac{\sigma_I}{E} - \nu\frac{\sigma_{II}}{E} - \nu\frac{\sigma_{III}}{E} = \frac{1+\nu}{E}\sigma_I - \frac{\nu}{E}(\sigma_I + \sigma_{II} + \sigma_{III})$$
$$\varepsilon_{II} = \frac{\sigma_{II}}{E} - \nu\frac{\sigma_I}{E} - \nu\frac{\sigma_{III}}{E} = \frac{1+\nu}{E}\sigma_{II} - \frac{\nu}{E}(\sigma_I + \sigma_{II} + \sigma_{III})$$
$$\varepsilon_{III} = \frac{\sigma_{III}}{E} - \nu\frac{\sigma_I}{E} - \nu\frac{\sigma_{II}}{E} = \frac{1+\nu}{E}\sigma_{III} - \frac{\nu}{E}(\sigma_I + \sigma_{II} + \sigma_{III})$$

i.e.

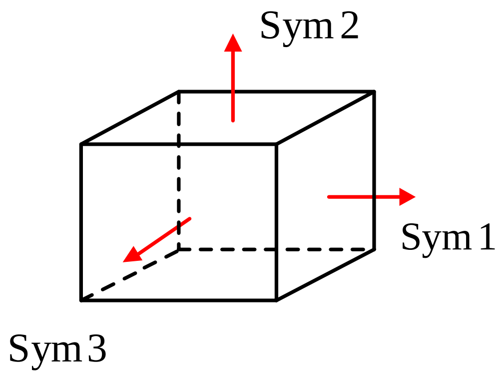
Therefore, we have

$$\underline{\varepsilon} = \frac{1+\nu}{E}\underline{\sigma} - \frac{\nu}{E}\sigma_{kk}\underline{I}$$

Through change of coordinates, this relation remains true for any other coordinate systems.

Orthotropic materials

An orthotropic material (e.g. wood, composites, fiber glass, etc) has three mutually perpendicular symmetry planes.

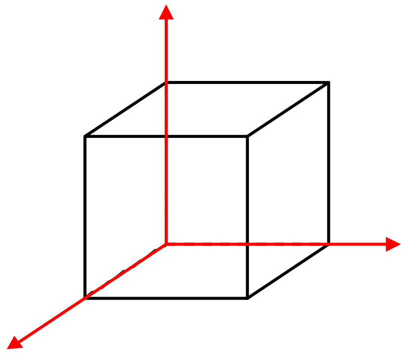


$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{pmatrix} = \begin{pmatrix} & & \\ & \mathbf{C}_{6\times 6} & \\ & & \end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{13}\\ \varepsilon_{12}\end{pmatrix}$$

$$\mathbf{C}: \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix} \text{(9 elastic constants)}$$

Cubic materials

Single crystal metals: FCC (Cu, Al, Ag, etc.); BCC (Fe, etc.)



$$\mathbf{C}:\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{44} & 0 \\ & & & & & C_{44} \end{pmatrix} \text{(3 elastic constants)}$$