Review of deformation tensors:

$$\underline{F},\ \underline{C},\ \underline{B},\ \underline{U},\ \underline{V},\ \underline{R},\ \underline{E},\ \underline{E}^*$$

Given $\underline{F}$, one can follow the following standard procedure to determine the other strain measures.

1) Most simply, $\underline{C} = \underline{F}^T \underline{F}$, $\underline{B} = \underline{F}\,\underline{F}^T$, $\underline{E} = \frac{1}{2}\left(\underline{C} - \underline{I}\right)$, $\underline{E}^* = \frac{1}{2}\left(\underline{I} - \underline{B}^{-1}\right)$

2) To fine $\underline{U}$, $\underline{V}$, $\underline{R}$, we need to perform the eigenvalue analysis of $\underline{C}$ and $\underline{B}$ (diagonalization of matrices):

$$\underline{C}\vec{m} = \lambda^2 \vec{m} \quad \Rightarrow \quad \underline{C} = \begin{pmatrix} \lambda_I^2 & 0 & 0 \\ 0 & \lambda_{II}^2 & 0 \\ 0 & 0 & \lambda_{III}^2 \end{pmatrix} = \lambda_I^2 \vec{m}_I \otimes \vec{m}_I + \lambda_{II}^2 \vec{m}_{II} \otimes \vec{m}_{II} + \lambda_{III}^2 \vec{m}_{III} \otimes \vec{m}_{III}$$

$$\underline{B}\vec{n} = \lambda^2 \vec{n} \quad \Rightarrow \quad \underline{B} = \begin{pmatrix} \lambda_I^2 & 0 & 0 \\ 0 & \lambda_{II}^2 & 0 \\ 0 & 0 & \lambda_{III}^2 \end{pmatrix} = \lambda_I^2 \vec{n}_I \otimes \vec{n}_I + \lambda_{II}^2 \vec{n}_{II} \otimes \vec{n}_{II} + \lambda_{III}^2 \vec{n}_{III} \otimes \vec{n}_{III}$$

3)

$$\underline{U} = \lambda_I \vec{m}_I \otimes \vec{m}_I + \lambda_{II} \vec{m}_{II} \otimes \vec{m}_{II} + \lambda_{III} \vec{m}_{III} \otimes \vec{m}_{III}$$

$$\underline{V} = \lambda_I \vec{n}_I \otimes \vec{n}_I + \lambda_{II} \vec{n}_{II} \otimes \vec{n}_{II} + \lambda_{III} \vec{n}_{III} \otimes \vec{n}_{III}$$

$$U_{ij} = \vec{e}_i \cdot \underline{U}\vec{e}_j,\ V_{ij} = \vec{e}_i \cdot \underline{V}\vec{e}_j \qquad \Rightarrow \qquad \underline{U} \text{ and } \underline{V}$$

4)

$$\underline{F} = \underline{R}\,\underline{U} = \underline{V}\,\underline{R}$$

$$\underline{R} = \underline{F}\,\underline{U}^{-1} = \underline{V}^{-1}\underline{F}$$

Generalized Hooke's law

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},\quad \mu = \frac{E}{2(1+\nu)}$$

Alternative forms in terms of deviatoric stresses/strains:

$$\sigma_{ij} = \sigma_{ij}^{'} + \frac{\sigma_{kk}}{3}\delta_{ij}$$

$$\varepsilon_{ij} = \varepsilon_{ij}^{'} + \frac{\varepsilon_{kk}}{3}\delta_{ij}$$

$$\varepsilon_{ij}^{'} + \frac{\varepsilon_{kk}}{3}\delta_{ij} = \frac{1+\nu}{E}\left(\sigma_{ij}^{'} + \frac{\sigma_{kk}}{3}\delta_{ij}\right) - \frac{\nu}{E}\sigma_{kk}\delta_{ij} = \frac{1+\nu}{E}\sigma_{ij}^{'} + \frac{1-2\nu}{3E}\sigma_{kk}\delta_{ij}$$

$$\varepsilon_{kk} = \frac{1-2\nu}{E}\sigma_{kk} = \frac{\sigma_{kk}}{3K}$$

$$\varepsilon_{ij}^{'} = \frac{1+\nu}{E}\sigma_{ij}^{'} = \frac{\sigma_{ij}^{'}}{2\mu}$$

$$\varepsilon_{ij} = \varepsilon_{ij}^{'} + \frac{\varepsilon_{kk}}{3}\delta_{ij} = \frac{\sigma_{ij}^{'}}{2\mu} + \frac{\sigma_{kk}}{9K}\delta_{ij}$$

Strain energy:

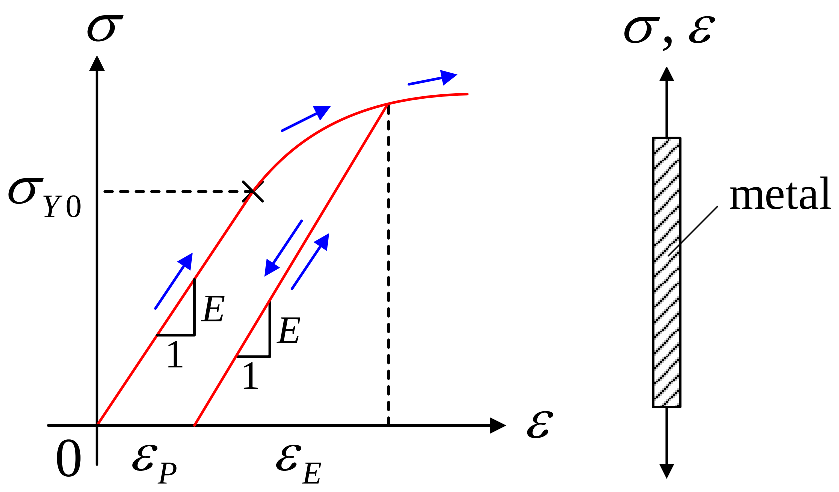
$$\mathrm{d}w = \sigma_{ij}\mathrm{d}\varepsilon_{ij}$$

For linear elastic solids,

$$\begin{aligned} w &= \int \sigma_{ij}\mathrm{d}\varepsilon_{ij} = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \\ &= \frac{1}{2}\left(\sigma_{ij}^{'} + \frac{\sigma_{kk}}{3}\delta_{ij}\right)\left(\varepsilon_{ij}^{'} + \frac{\varepsilon_{kk}}{3}\delta_{ij}\right) \\ &= \frac{1}{2}\sigma_{ij}^{'}\varepsilon_{ij}^{'} + \frac{1}{6}\sigma_{kk}\varepsilon_{kk} \end{aligned}$$

$$w = \mu\varepsilon_{ij}^{'}\varepsilon_{ij}^{'} + \frac{1}{2}K\varepsilon_{kk}^{2} \quad \text{or} \quad w = \frac{1}{4\mu}\sigma_{ij}^{'}\sigma_{ij}^{'} + \frac{1}{18K}\sigma_{kk}^{2}$$

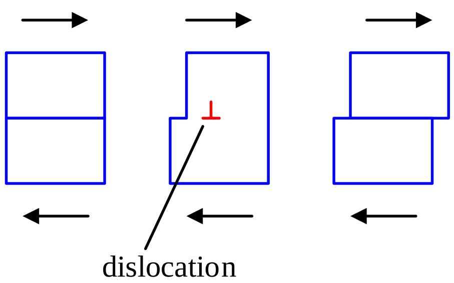
Plastic material behavior



Important experimental facts:

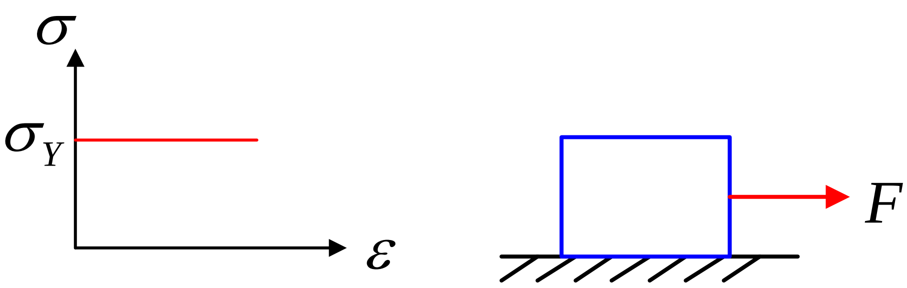
1) Hydrostatic stress has no effects on plastic deformation;
2) Plastic behavior doesn't induce volume change of a material.

In 1930's, Taylor and other scientists discovered that plastic deformation is caused by shearing of atomic planes via propagation of a type of lattice defects called dislocations.

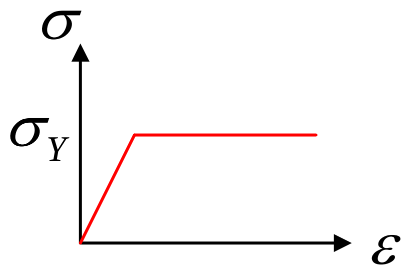


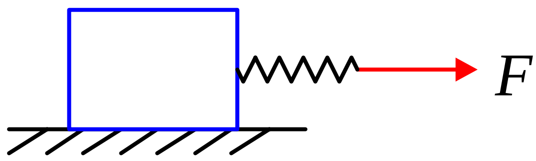
1D models:

Rigid-Perfectly plastic material

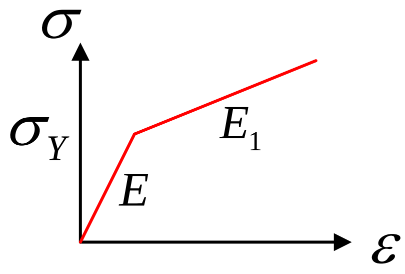


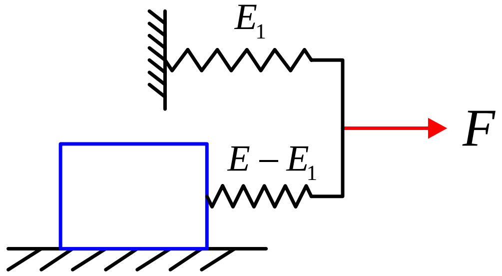
Elastic-perfectly plastic material



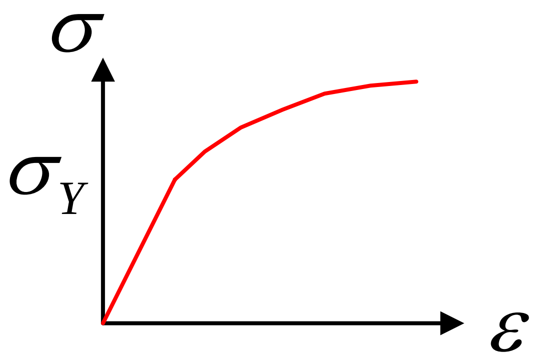


Linear hardening material





Power law hardening material



$$\sigma = \begin{cases} E\varepsilon, & \sigma < \sigma_Y \\ \sigma_Y\left(1+\dfrac{E\varepsilon_P}{\sigma_Y}\right)^N, & \sigma \ge \sigma_Y \end{cases}$$

Here σ_Y and N can be treated as fitting parameters to experimental data; $0 \le N < 1$ is called the hardening index.

In incremental form:

$$d\varepsilon_P = \begin{cases} \dfrac{d\sigma}{h}, & \text{plastic loading} \\ 0, & \text{otherwise} \end{cases}$$

Here $h = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon_P} = EN\left(1 + \frac{E\varepsilon_P}{\sigma_Y}\right)^{N-1}$ is the tangent modulus of the curve of stress versus plastic strain.

The total strain increment can be decomposed as an elastic and a plastic part as

$$\mathrm{d}\varepsilon = \mathrm{d}\varepsilon_E + \mathrm{d}\varepsilon_P = \frac{\mathrm{d}\sigma}{E} + \mathrm{d}\varepsilon_P$$

How to generalize to 3D?

1) Yield criterion
2) Plastic constitutive law ($\sigma - \varepsilon$ relation)

Yield criterion

For 1D,

$$\sigma = \sigma_Y$$

For 3D,
Tresca condition: maximum shear stress = critical $\Rightarrow$ Yield

If $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$,

$$\frac{\sigma_I - \sigma_{III}}{2} = C_Y = \frac{\sigma_Y}{2} \Rightarrow \sigma_I - \sigma_{III} = \sigma_Y$$

For general $\sigma_I, \sigma_{II}, \sigma_{III}$,

$$\mathrm{Max}\left(\left|\sigma_I - \sigma_{II}\right|, \left|\sigma_I - \sigma_{III}\right|, \left|\sigma_{II} - \sigma_{III}\right|\right) = \sigma_Y$$

Von Mises condition:

$$w = \frac{1}{4\mu}\sigma'_{ij}\sigma'_{ij} + \frac{1}{18K}\sigma_{kk}^2$$

If we take the distortional part of elastic energy as a criterion for the onset of plastic deformation, we can write the yield condition as

$$\sigma'_{ij}\sigma'_{ij} = C_Y$$

In principal stress orientations,

$$\sigma'_{ij}\sigma'_{ij} = {\sigma'_I}^2 + {\sigma'_{II}}^2 + {\sigma'_{III}}^2$$

$$\sigma'_I = \sigma_I - \frac{1}{3}\left(\sigma_I + \sigma_{II} + \sigma_{III}\right) = \frac{1}{3}\left(2\sigma_I - \sigma_{II} - \sigma_{III}\right)$$

Similarly,

$$\sigma_{II}^{'} = \frac{1}{3}\left(2\sigma_{II} - \sigma_{I} - \sigma_{III}\right)$$

$$\sigma_{III}^{'} = \frac{1}{3}\left(2\sigma_{III} - \sigma_{I} - \sigma_{II}\right)$$

$$\sigma_{ij}^{'}\sigma_{ij}^{'} = \frac{1}{9}\left(\left(2\sigma_{I} - \sigma_{II} - \sigma_{III}\right)^2 + \left(2\sigma_{II} - \sigma_{I} - \sigma_{III}\right)^2 + \left(2\sigma_{III} - \sigma_{I} - \sigma_{II}\right)^2\right)$$

$$= \frac{2}{3}\left(\sigma_I^2 + \sigma_{II}^2 + \sigma_{III}^2 - \sigma_I\sigma_{II} - \sigma_{II}\sigma_{III} - \sigma_I\sigma_{III}\right)$$

Specify to 1D,

$$\frac{2}{3}\sigma^2 = C_Y = \frac{2}{3}\sigma_Y^2 \Rightarrow \sigma_{ij}^{'}\sigma_{ij}^{'} = \frac{2}{3}\sigma_Y^2$$

i.e.

$$\sigma_e = \sqrt{\frac{3}{2}\sigma_{ij}^{'}\sigma_{ij}^{'}} = \sigma_Y$$

which is called the von Mises stress.