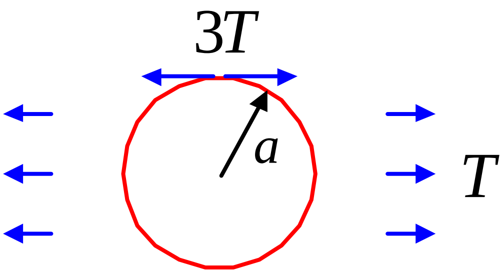
Continue on the problem of circular hole under uniaxial tension (remote).

Stress concentration occurs at $r = a$, $\theta = \frac{\pi}{2}$.



Governing equation is: $\nabla^2 \nabla^2 \phi = 0$

The stress components in polar coordinates are:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

Boundary conditions are:

@ $r = a$, $\sigma_{rr} = \sigma_{r\theta} = 0$

@ $r = \infty$, $\underline{\sigma} = T\vec{e}_1 \otimes \vec{e}_1$

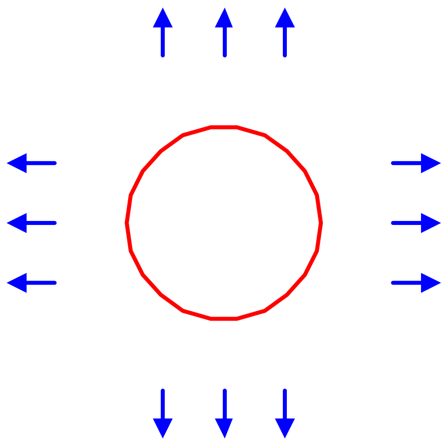
$$\sigma_{rr} = \vec{e}_r \cdot \underline{\sigma}\vec{e}_r = \frac{T}{2}(1+\cos 2\theta)$$

$$\sigma_{\theta\theta} = \frac{T}{2}(1-\cos 2\theta)$$

$$\sigma_{r\theta} = -\frac{T}{2}\sin 2\theta$$

Therefore, the boundary condition at infinity can be decomposed into two parts.

Part I:



@ $r=\infty$, $\sigma_{rr}=\sigma_{\theta\theta}=\dfrac{T}{2}$. For this part, we have previously obtained the solution as

$$\sigma_{rr}=\frac{T}{2}\left(1-\frac{a^2}{r^2}\right)$$

$$\sigma_{\theta\theta}=\frac{T}{2}\left(1+\frac{a^2}{r^2}\right)$$

$$\sigma_{r\theta}=0$$

Part II:

@ $r=\infty$, $\sigma_{rr}=\dfrac{T}{2}\cos 2\theta$, $\sigma_{\theta\theta}=-\dfrac{T}{2}\cos 2\theta$, $\sigma_{r\theta}=-\dfrac{T}{2}\sin 2\theta$

These expressions suggests $\phi=f(r)\cos 2\theta$. Inserting it into $\nabla^2\nabla^2\phi=0$ gives

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{4}{r^2}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{4}{r^2}\right)f(r)=0$$

Assume: $f(r)=r^\lambda \Rightarrow \lambda(\lambda-4)(\lambda-2)(\lambda+2)=0 \Rightarrow \lambda=0, 2, -2, 4$

$$f(r)=C_1r^2+C_2r^4+C_3r^{-2}+C_4$$

Using boundary conditions:

@ $r=a$, $\sigma_{rr}=\sigma_{r\theta}=0$

@ $r=\infty$, $\sigma_{rr}=\dfrac{T}{2}\cos 2\theta$, $\sigma_{r\theta}=-\dfrac{T}{2}\sin 2\theta$

We can determine the constant coefficients as

$$C_1 = -\frac{1}{4}T, \quad C_2 = 0, \quad C_3 = -\frac{1}{4}a^4 T, \quad C_4 = \frac{1}{2}a^2 T$$

Adding the solution to part I, the complete solutions of stress components are:

$$\sigma_{rr} = \frac{T}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{T}{2}\left(1 - \frac{a^2}{r^2}\right)\left(1 - \frac{3a^2}{r^2}\right)\cos 2\theta$$

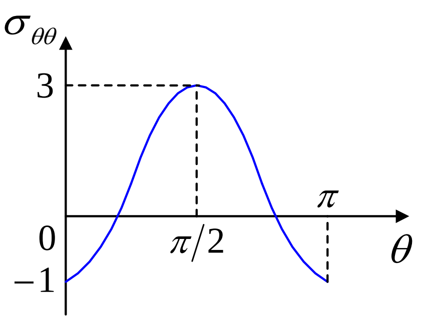
$$\sigma_{\theta\theta} = \frac{T}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{T}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

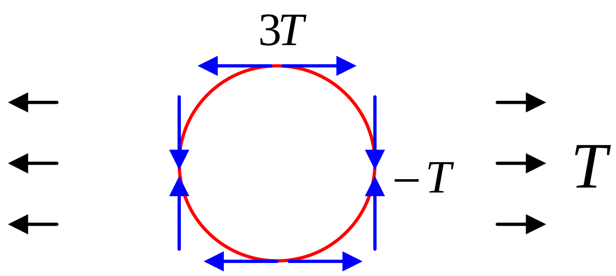
$$\sigma_{r\theta} = -\frac{T}{2}\left(1 - \frac{a^2}{r^2}\right)\left(1 + \frac{3a^2}{r^2}\right)\sin 2\theta$$

The hoop stress @ $r = a$

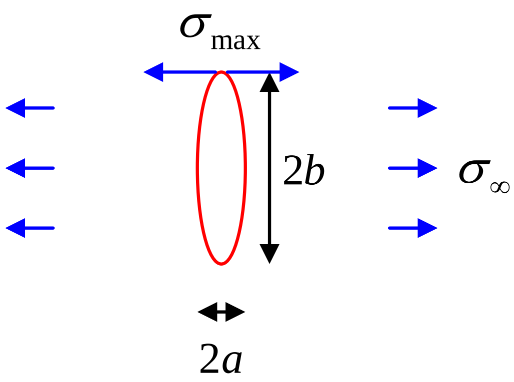
$$\sigma_{\theta\theta} = T(1 - 2\cos 2\theta)$$

has the maximum at $\theta = \frac{\pi}{2}$.





For an elliptic hole,

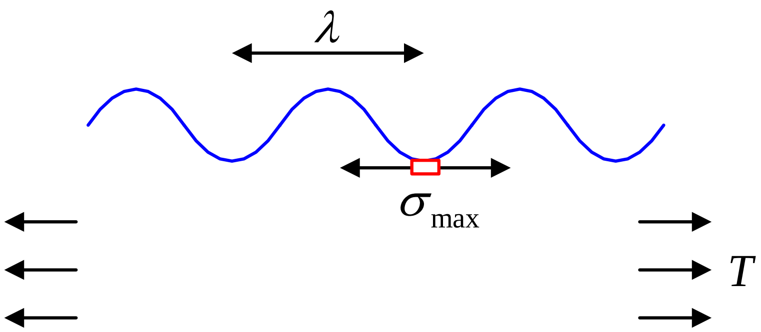


$$\sigma_{max} = T\left(1 + 2\frac{b}{a}\right)$$

Radius of curvature at the end of semi-major axis is: $\rho = \frac{a^2}{b}$. We can rewrite this solution as

$$\sigma_{\max} = T\left(1 + 2\sqrt{\frac{b}{\rho}}\right)$$

If $\rho = b$, $\sigma_{\max} = 3T$, the result reduces to the case of circular hole. The above behavior is fairly typical of stress near a groove or hole. For example, consider the stress concentration at a slightly wavy surface under tension.



The surface has a profile of

$$y = A \sin \frac{2\pi x}{\lambda}$$

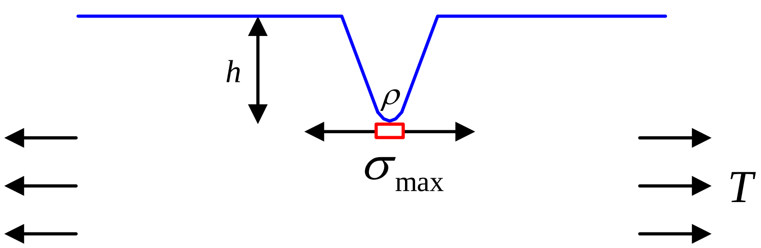
The local curvature at a surface valley is

$$\left.\frac{1}{\rho}\right|_{\text{valley}} = y'' = A\left(\frac{2\pi}{\lambda}\right)^2$$

Recalling the result for the maximum stress at the valley, we can write

$$\sigma_{\max} = T\left(1 + 4\pi\frac{A}{\lambda}\right) = T\left(1 + 2\sqrt{\frac{A}{\rho}}\right)$$

This has the same form as that near an elliptical hole. These results suggest that stress concentration occurs at places with negative curvature (concave spots of a material/structure). For a general crack/notch under tension,

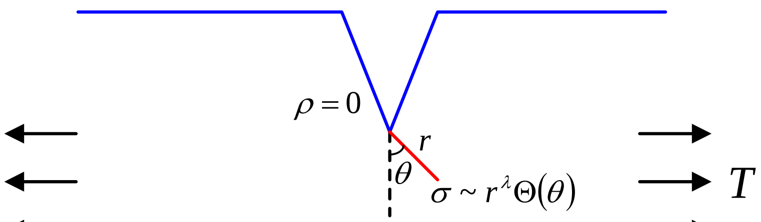


the maximum stress occurs at the crack/notch tip can be expressed as

$$\sigma_{\max} = T\left(1+\alpha\sqrt{\frac{h}{\rho}}\right)$$

where ρ is the radius of curvature at the tip, h is the depth of the notch, and α is a geometric factor (equal to 2 for an elliptical hole).

Remark:



We note that for crack-like flaws, $\sigma_{\max} \to \infty$ when $\rho \to 0$, which presents a challenge for failure analysis. Fracture mechanics developed in the mid-20th century shows that elasticity solutions for such flaws generally have the form of $\sigma = Kr^{-\frac{1}{2}}\Theta(\theta)$.

$$\phi \sim r^{\lambda+2} f(\theta)$$

$$\sigma \sim r^{\lambda}\Theta(\theta)$$

$$\lambda = -\frac{1}{2} \text{ for cracks}$$

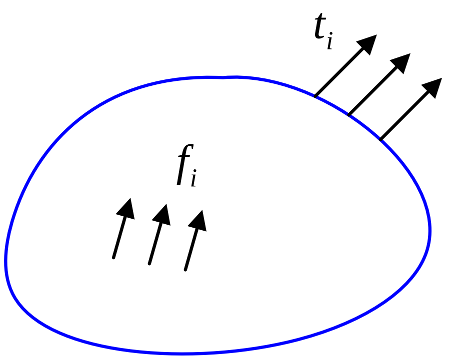
$$-\frac{1}{2} < \lambda \le 0 \text{ for notches}$$

The coefficient K is called the stress intensity factor. For such sharp cracks/notches, stress itself is no longer a useful criterion, rather the coefficient of the singularity, K, turns out to be the appropriate quantity for the behavior of cracks/notches.

Failure criterion: $K \le K_C$, where K_C is a material property called fracture toughness.

In contrast, the classical failure criterion based on strength of material has the form $\sigma \le \sigma_C$, which is clearly inappropriate as it predicts materials have no resistance to sharp cracks.

Chap. 7 Variational/energy methods in elastic solids



Principle of virtual work

$$\int_V f_i \delta u_i \mathrm{d}V + \int_S t_i \delta u_i \mathrm{d}S = \int_V \delta w \mathrm{d}V$$
$$\Rightarrow \delta\left(\int_V w \mathrm{d}V - \int_V f_i u_i \mathrm{d}V - \int_S t_i u_i \mathrm{d}S\right) = 0 \Rightarrow \delta V = 0$$

Here $V = \int_V w \mathrm{d}V - \int_V f_i u_i \mathrm{d}V - \int_S t_i u_i \mathrm{d}S$ is the total potential energy of the system. The first term is the strain energy stored in the elastic body,

$$w = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

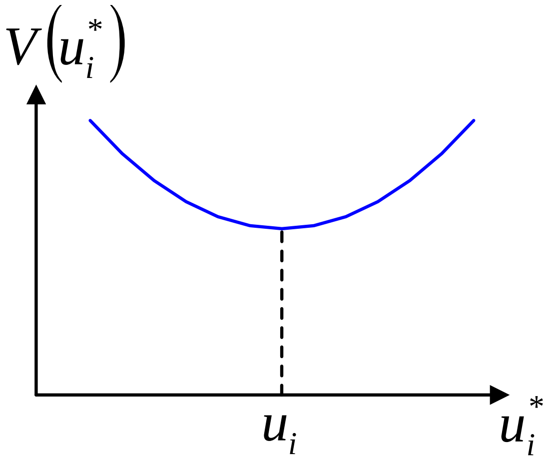
Principle of virtual work shows that V is stationary. In fact, V is minimum with respect to variational displacement. If u_i is the actual displacement field, then $u_i^* = u_i + \delta u_i$ would always increase V.

Proof of the principle of minimum potential energy:

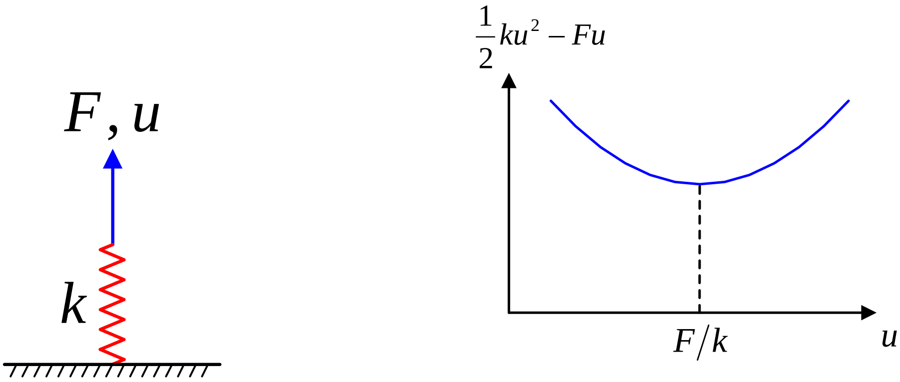
Consider a kinematically admissible displacement field $u_i^* = u_i + \delta u_i$ (a field satisfying all displacement BCs but not necessarily the actual solution).

$$\begin{aligned}
V\left(u_i^*\right) &= \int_V \frac{1}{2} C_{ijkl}\varepsilon_{ij}^*\varepsilon_{kl}^* \mathrm{d}V - \int_V f_i u_i^* \mathrm{d}V - \int_S t_i u_i^* \mathrm{d}S \\
&= \int_V \frac{1}{2} C_{ijkl}\left(\varepsilon_{ij} + \delta\varepsilon_{ij}\right)\left(\varepsilon_{kl} + \delta\varepsilon_{kl}\right)\mathrm{d}V - \int_V f_i\left(u_i + \delta u_i\right)\mathrm{d}V - \int_S t_i\left(u_i + \delta u_i\right)\mathrm{d}S \\
&= V\left(u_i\right) + \int_V \delta w \mathrm{d}V - \int_V f_i \delta u_i \mathrm{d}V + \int_S t_i \delta u_i \mathrm{d}S + \frac{1}{2}\int_V C_{ijkl}\delta\varepsilon_{ij}\delta\varepsilon_{kl}\mathrm{d}V \\
&= V\left(u_i\right) + \frac{1}{2}\int_V C_{ijkl}\delta\varepsilon_{ij}\delta\varepsilon_{kl}\mathrm{d}V \geq V\left(u_i\right)
\end{aligned}$$

(The second term is always positive due to the positive definiteness of elastic modulus).



Simple 1D analog:



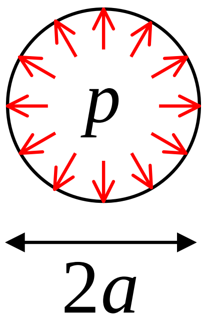
Consider a 1D linear spring under applied force, the potential energy of the system is

$$V(u) = \frac{1}{2}ku^2 - Fu$$

Minimum potential energy requires that

$$\frac{\partial V}{\partial u} = ku - F = 0 \Rightarrow u = \frac{F}{k}$$

Example: Pressurized hole in an infinite elastic body



The pressurized hole should not disturb material at infinity. Take the simplest decay function about the displacement,

$$u_r^* = \alpha \frac{1}{r}$$

Based on the assumed displacement field, the strain components are

$$\varepsilon_{rr}^* = \frac{\partial u_r^*}{\partial r} = -\alpha \frac{1}{r^2}$$

$$\varepsilon_{\theta\theta}^* = \frac{u_r^*}{r} = \alpha \frac{1}{r^2}$$

$$\varepsilon_{r\theta}^* = 0$$

Using Hooke's law, the stresses are

$$\sigma_{rr}^* = -\frac{E}{1+\nu}\frac{\alpha}{r^2}$$

$$\sigma_{\theta\theta}^* = \frac{E}{1+\nu}\frac{\alpha}{r^2}$$

$$\sigma_{r\theta}^* = 0$$

The strain energy density can thus be calculated as

$$w = \frac{1}{2}\left(\sigma_{rr}^*\varepsilon_{rr}^* + \sigma_{\theta\theta}^*\varepsilon_{\theta\theta}^*\right) = \frac{E}{1+\nu}\frac{\alpha^2}{r^4}$$

The potential energy of the system is

$$V = \int_a^\infty \mathrm{d}r \int_0^{2\pi} r\mathrm{d}\theta \frac{E}{1+\nu}\frac{\alpha^2}{r^4} - \int_0^{2\pi} a\mathrm{d}\theta\, p\frac{\alpha}{a}$$
$$= 2\pi\left(\frac{E}{2(1+\nu)}\frac{\alpha^2}{a^2} - p\alpha\right)$$

To minimize $V(\alpha)$, we must have

$$\frac{\partial V}{\partial \alpha} = 0 \Rightarrow \frac{E}{1+\nu}\frac{\alpha}{a^2} - p = 0 \Rightarrow \alpha = \frac{(1+\nu)pa^2}{E}$$

Once the parameter α is determined, we can write out the complete solution of the problem:

$$u_r^* = \frac{(1+\nu)p}{E}\frac{a^2}{r}$$

$$\varepsilon_{rr}^* = \frac{\partial u_r^*}{\partial r} = -\frac{(1+\nu)p}{E}\frac{a^2}{r^2}, \quad \varepsilon_{\theta\theta}^* = \frac{(1+\nu)p}{E}\frac{a^2}{r^2}$$

$$\sigma_{rr}^* = -p\frac{a^2}{r^2}, \quad \sigma_{\theta\theta}^* = p\frac{a^2}{r^2}$$

It happens that our simple guess about displacement hits the exact solution of the present problem.