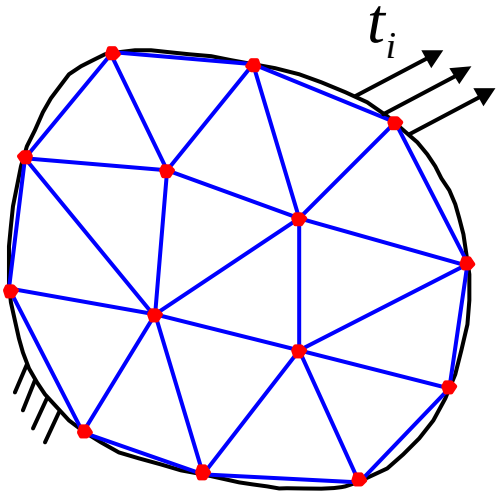
Chap. 9 Finite element method (Read Chap 7 of Prof Bower's notes)

Principle of virtual work:

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V = \int_V f_i\delta u_i\mathrm{d}V + \int_{A_T} t_i\delta u_i\mathrm{d}S$$

Principle of minimum potential energy

$$\mathrm{Min}\left(V = \int_V \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\mathrm{d}V - \int_V f_i u_i\mathrm{d}V - \int_{A_T} t_i u_i\mathrm{d}S\right)$$



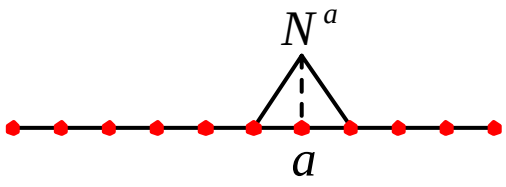
We can represent the displacement by interpolation through its values at a network of nodes (discretization),

$$\vec{u}(\vec{x}) = \sum_{a=1}^{n} N^a(\vec{x})\vec{u}^a = N^a(\vec{x})\vec{u}^a$$

where n is the number of FEM nodes, $\vec{u}^a$ is the nodal displacement, and $N^a(\vec{x})$ are the interpolation functions.

$$N^a(\vec{x}) = \begin{cases} 1, & \text{if } \vec{x} = \vec{x}^a \\ 0, & \text{at all other nodes} \end{cases}$$

Recall such interpolation function for 1D element as follows.



More non-local forms of N^a have been proposed in meshless/element-free FEM methods.

FEM formulation based on principle of virtual work

Consider a virtual displacement field: $\delta\vec{u}(\vec{x}) = N^b(\vec{x})\delta\vec{u}^b$ (Note summation convention over repeated indices).

Inserting $\vec{u}(\vec{x}) = N^a\vec{u}^a$, $\delta\vec{u}(\vec{x}) = N^b(\vec{x})\delta\vec{u}^b$ into Principle of Virtual Work, the left side of the equation becomes

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V = \int_V C_{ijkl}\varepsilon_{kl}\delta\varepsilon_{ij}\mathrm{d}V = \int_V C_{ijkl}u_{k,l}\delta u_{i,j}\mathrm{d}V = \int_V C_{ijkl}\frac{\partial N^a}{\partial x_l}\frac{\partial N^b}{\partial x_j}u_k^a\delta u_i^b\mathrm{d}V$$

$$= K_{aibk}u_k^a\delta u_i^b$$

The right side of the equation becomes

$$\int_V f_i\delta u_i\mathrm{d}V + \int_{A_T} t_i\delta u_i\mathrm{d}S = \left(\int_V f_i N^b\mathrm{d}V + \int_{A_T} t_i N^b\mathrm{d}S\right)\delta u_i^b$$

$$= F_i^b\delta u_i^b$$

The principle of virtual work becomes

$$\left(K_{aibk}u_k^a - F_i^b\right)\delta u_i^b = 0$$

Since this must be true for any δu_i^b,

$$K_{aibk}u_k^a - F_i^b = 0$$

In matrix form, this is a set of linear algebraic equations:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & \cdots & \cdots & K_{1,3n} \\ K_{21} & K_{22} & K_{23} & \cdots & \cdots & \cdots & K_{2,3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{3n,1} & K_{3n,2} & K_{3n,3} & \cdots & \cdots & \cdots & K_{3n,3n} \end{bmatrix}_{3n\times 3n} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_3^{(2)} \\ \vdots \end{bmatrix}_{3n\times 1} = \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ \vdots \end{bmatrix}_{3n\times 1}$$

or

$$[\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{F}\}$$

FEM formulation based on principle of minimum potential energy

$$\int_V \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\mathrm{d}V = \int_V \frac{1}{2}C_{ijkl}u_{i,j}u_{k,l}\mathrm{d}V = \frac{1}{2}u_i^a u_k^b \int_V C_{ijkl}\frac{\partial N^a}{\partial x_j}\frac{\partial N^b}{\partial x_l}\mathrm{d}V$$
$$= \frac{1}{2}\{\mathbf{U}\}^T[\mathbf{K}]\{\mathbf{U}\}$$

$$\int_V f_i u_i \mathrm{d}V + \int_{A_T} t_i u_i \mathrm{d}S = \left(\int_V f_i N^a \mathrm{d}V + \int_{A_T} t_i N^a \mathrm{d}S\right)u_i^a$$
$$= \{\mathbf{U}\}^T\{\mathbf{F}\}$$

where $[\mathbf{K}]$, $\{\mathbf{U}\}$, $\{\mathbf{F}\}$ are as defined in the previous page. The potential energy of the system can thus be written as

$$V = \frac{1}{2}\{\mathbf{U}\}^T[\mathbf{K}]\{\mathbf{U}\} - \{\mathbf{U}\}^T\{\mathbf{F}\}$$

Minimizing V requires

$$\frac{\partial V}{\partial u_i^a} = 0 \text{ for all } a = 1, 2, \cdots, n,\ i = 1, 2, 3$$

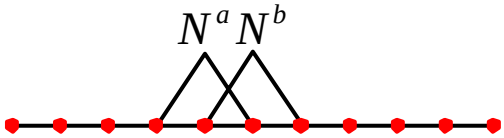
which leads to the same FEM equation:

$$[\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{F}\}$$

Comment:

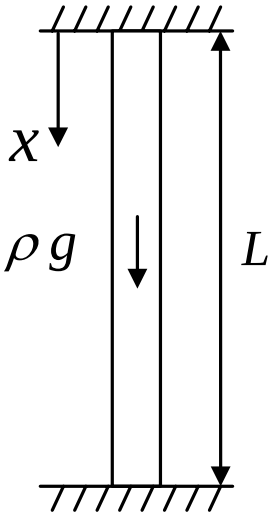
$[\mathbf{K}]$ is a sparse matrix since $\int_V C_{ijkl}\frac{\partial N^a}{\partial x_j}\frac{\partial N^b}{\partial x_l}\mathrm{d}V = 0$ unless a and b are close to each other.

$$\int_V C_{ijkl}\frac{\partial N^a}{\partial x_j}\frac{\partial N^b}{\partial x_l}\mathrm{d}V = 0$$



$$\int_V C_{ijkl}\frac{\partial N^a}{\partial x_j}\frac{\partial N^b}{\partial x_l}\mathrm{d}V \neq 0$$

Recall the 1D problem:



$$K_{ab} = \int_V E \frac{\partial N^a}{\partial x} \frac{\partial N^b}{\partial x} \mathrm{d}V = S \int_0^L E \frac{\partial N^a}{\partial x} \frac{\partial N^b}{\partial x} \mathrm{d}x$$

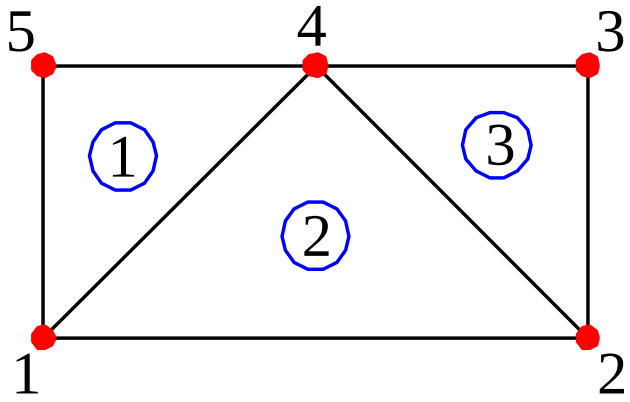
$$F^a = \int_V f N^a \mathrm{d}V = S \int_0^L f N^a \mathrm{d}x$$

where S is the cross section area of the bar. In this case, the FEM equation becomes

$$[\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{F}\} \Rightarrow \int_0^L E \frac{\partial N^a}{\partial x} \frac{\partial N^b}{\partial x} \mathrm{d}x\, u^b = \int_0^L f N^a \mathrm{d}x$$

which is the same equation used in the beginning of the class.

Implementation of FEM in 2D solids



In the above configuration, we can identify the following concepts:

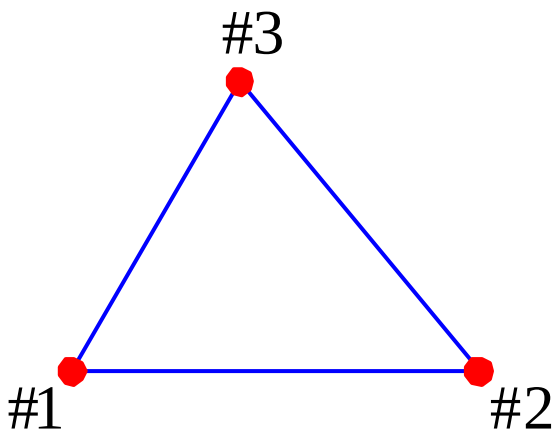
FEM nodes: 1, 2, 3, 4, 5

Elements: ①, ②, ③

Nodal coordinates:

1	-1	0
2	1	0
3	1	1
4	0	1
5	-1	1

Local nodes of a generic element:



Element connectivity table:

Element No.	#1	#2	#3
①	1	4	5
②	1	2	4
③	2	3	4

Assume linear triangular element (linear interpolation function):

$$N^1(x_1,x_2)=\frac{\left(x_2-x_2^{\#2}\right)\left(x_1^{\#3}-x_1^{\#2}\right)-\left(x_1-x_1^{\#2}\right)\left(x_2^{\#3}-x_2^{\#2}\right)}{\left(x_2^{\#1}-x_2^{\#2}\right)\left(x_1^{\#3}-x_1^{\#2}\right)-\left(x_1^{\#1}-x_1^{\#2}\right)\left(x_2^{\#3}-x_2^{\#2}\right)}$$

$$N^2(x_1,x_2)=\frac{\left(x_2-x_2^{\#3}\right)\left(x_1^{\#1}-x_1^{\#3}\right)-\left(x_1-x_1^{\#3}\right)\left(x_2^{\#1}-x_2^{\#3}\right)}{\left(x_2^{\#2}-x_2^{\#3}\right)\left(x_1^{\#1}-x_1^{\#3}\right)-\left(x_1^{\#2}-x_1^{\#3}\right)\left(x_2^{\#1}-x_2^{\#3}\right)}$$

$$N^3(x_1,x_2)=\frac{(x_2-x_2^{\#1})(x_1^{\#2}-x_1^{\#1})-(x_1-x_1^{\#1})(x_2^{\#2}-x_2^{\#1})}{(x_2^{\#3}-x_2^{\#1})(x_1^{\#2}-x_1^{\#1})-(x_1^{\#3}-x_1^{\#1})(x_2^{\#2}-x_2^{\#1})}$$

$$\bar{u}(x_1,x_2)=\sum_{a=1}^{3}N^a(x_1,x_2)\bar{u}^a=N^1\bar{u}^{\#1}+N^2\bar{u}^{\#2}+N^3\bar{u}^{\#3}$$

The strain vector can be expressed in terms of local nodal displacements as:

$$\underline{\varepsilon}=\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}=\begin{bmatrix}\partial u_1/\partial x_1\\ \partial u_2/\partial x_2\\ \partial u_1/\partial x_2+\partial u_2/\partial x_1\end{bmatrix}=\begin{bmatrix}\frac{\partial N^1}{\partial x_1} & 0 & \frac{\partial N^2}{\partial x_1} & 0 & \frac{\partial N^3}{\partial x_1} & 0\\ 0 & \frac{\partial N^1}{\partial x_2} & 0 & \frac{\partial N^2}{\partial x_2} & 0 & \frac{\partial N^3}{\partial x_2}\\ \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^3}{\partial x_2} & \frac{\partial N^3}{\partial x_1}\end{bmatrix}\begin{bmatrix}u_1^{\#1}\\ u_2^{\#1}\\ u_1^{\#2}\\ u_2^{\#2}\\ u_1^{\#3}\\ u_2^{\#3}\end{bmatrix}=\underline{\underline{B}}\,\underline{u}_{el}$$

For plane stress problems, the constitutive law is

$$\sigma_{11}=\frac{E}{1-\nu^2}(\varepsilon_{11}+\nu\varepsilon_{22})$$
$$\sigma_{22}=\frac{E}{1-\nu^2}(\varepsilon_{22}+\nu\varepsilon_{11})$$
$$\sigma_{12}=\frac{E}{1+\nu}\varepsilon_{12}$$

This can be written as,

$$\underline{\sigma}=\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}=\underline{\underline{D}}\,\underline{\underline{B}}\,\underline{u}_{el}$$

The strain energy density is

$$w=\frac{1}{2}\sigma_{ij}\varepsilon_{ij}=\frac{1}{2}(\sigma_{11}\varepsilon_{11}+\sigma_{22}\varepsilon_{22}+\sigma_{12}\varepsilon_{12}+\sigma_{21}\varepsilon_{21})=\frac{1}{2}\begin{bmatrix}\sigma_{11} & \sigma_{22} & \sigma_{12}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}=\frac{1}{2}\underline{\sigma}^T\,\underline{\varepsilon}$$

The total strain energy in the element is obtained from integration

$$U_{el}=\int_{V_{el}}\frac{1}{2}\underline{\sigma}^T\,\underline{\varepsilon}\,\mathrm{d}V=\frac{1}{2}A_{el}\,\underline{\sigma}^T\,\underline{\varepsilon}=\frac{1}{2}A_{el}\,\underline{u}_{el}^T\,\underline{\underline{B}}^T\,\underline{\underline{D}}\,\underline{\underline{B}}\,\underline{u}_{el}=\frac{1}{2}\underline{u}_{el}^T\left[A_{el}\,\underline{\underline{B}}^T\,\underline{\underline{D}}\,\underline{\underline{B}}\right]\underline{u}_{el}=\frac{1}{2}\underline{u}_{el}^T\,\underline{K}_{el}\,\underline{u}_{el}$$

where $\underline{K}_{el}=A_{el}\,\underline{\underline{B}}^T\,\underline{\underline{D}}\,\underline{\underline{B}}$ is called the element stiffness matrix.

The total Stain energy in the entire body is the sum of strain energy in all elements,

$$U = \sum_{elements} U_{el} = \sum_{elements} \frac{1}{2} \underline{u}_{el}^T \underline{K}_{el} \underline{u}_{el} = \frac{1}{2} \underline{u}^T \underline{K} \underline{u}$$

where $\underline{u}$ and $\underline{K}$ are the global displacement vector and global stiffness matrix.

The question is how to assemble global displacement, stiffness matrix and nodal force from their counterparts on the element level. For this we use the connectivity table:

$$(\#1, \#2, \#3) \Rightarrow (a, b, c)$$

$$\underline{u}_{el} = \begin{Bmatrix} u_1^{\#1} \\ u_2^{\#1} \\ u_1^{\#2} \\ u_2^{\#2} \\ u_1^{\#3} \\ u_2^{\#3} \end{Bmatrix}_{6\times 1} \Rightarrow \begin{Bmatrix} \vdots \\ u_{2a-1} \\ u_{2a} \\ \vdots \\ u_{2b-1} \\ u_{2b} \\ \vdots \\ u_{2c-1} \\ u_{2c} \\ \vdots \end{Bmatrix}_{2n\times 1}$$

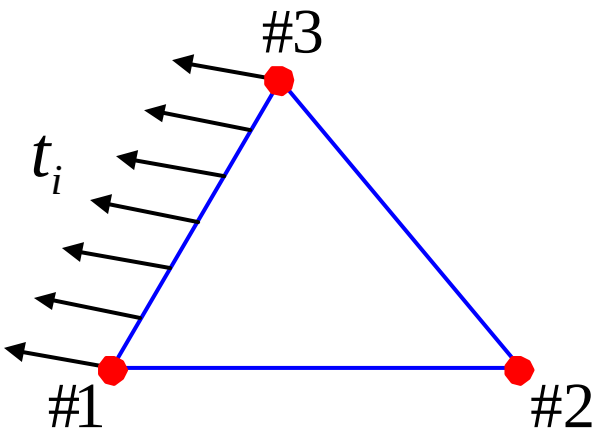
For stiffness entry:

$$\alpha(1, 2, 3, 4, 5, 6) \Rightarrow z_\alpha(2a-1, 2a, 2b-1, 2b, 2c-1, 2c)$$

We can thus assemble stiffness matrix according to the connectivity rule:

$$K_{z_\alpha z_\beta} = K_{z_\alpha z_\beta} + K_{\alpha\beta}^{el} \text{ (assembly of global stiffness matrix over all the elements)}$$

Assembly of nodal force:



For the element shown above, the potential energy of the applied traction is:

$$P^{element} = -\int_0^L t_i u_i \mathrm{d}s$$

Using linear interpolation functions, the displacement can be written as

$$u_i = u_i^{\#1}\frac{s}{L} + u_i^{\#3}\left(1-\frac{s}{L}\right)$$

$$P^{element} = -\int_0^L t_i(s)\frac{s}{L}\mathrm{d}s\, u_i^{\#1} - \int_0^L t_i(s)\left(1-\frac{s}{L}\right)\mathrm{d}s\, u_i^{\#3} = \underline{F}_{el}\,\underline{u}_{el}$$

Assembly into global nodal force vector:

$$\underline{F}_{el} = \begin{Bmatrix} F_1^{\#1} \\ F_2^{\#1} \\ F_1^{\#2} \\ F_2^{\#2} \\ F_1^{\#3} \\ F_2^{\#3} \end{Bmatrix}_{6\times 1} \Rightarrow \begin{Bmatrix} \vdots \\ F_{2a-1} \\ F_{2a} \\ \vdots \\ F_{2b-1} \\ F_{2b} \\ \vdots \\ F_{2c-1} \\ F_{2c} \\ \vdots \end{Bmatrix}_{2n\times 1}$$

or

$$\alpha(1,2,3,4,5,6) \Rightarrow z_\alpha(2a-1,\, 2a,\, 2b-1,\, 2b,\, 2c-1,\, 2c)$$

$$F_{z_\alpha} = F_{z_\alpha} + F_\alpha^{el}$$

After the global matrices are assembled, the FEM problem is to solve the linear algebraic equation

$$\underline{K}_{2n\times 2n}\,\underline{u}_{2n\times 1} = \underline{F}_{2n\times 1}$$