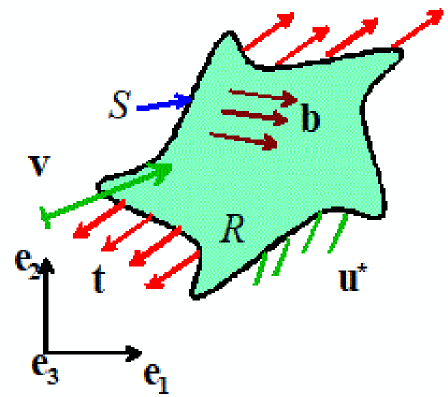
Review: Principle of Minimum Potential Energy



Assumptions:

1. Elastic material
2. Small displacements
3. Static equilibrium
4. Boundary conditions $\mathbf{u} = \mathbf{u}^*$ on S_1 $\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{t}$ on S_2

Definitions:

1. Kinematically admissible displacement field: any differentiable displacement vector satisfying $\mathbf{v} = \mathbf{u}^*$ wherever displacements are known
2. Actual displacement field (the one that satisfies equilibrium within the solid and traction boundary conditions on surfaces) $\mathbf{u}$
3. Strain energy density U
4. Potential energy

$$\Pi(\mathbf{v}) = \int_V \mathrm{U}(\mathbf{v}) dV - \int_V \mathbf{b} \cdot \mathbf{v} dV - \int_{S2} \mathbf{t} \cdot \mathbf{v} dA$$

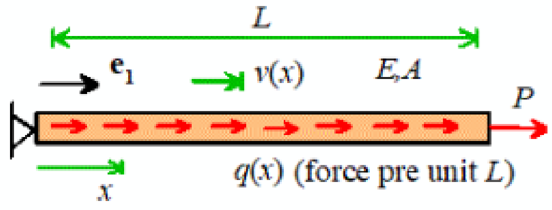
Principle of minimum potential energy $\Pi(\mathbf{v}) \geq \Pi(\mathbf{u})$

Among all guesses for the displacement field, the best guess is the one with the smallest Π

Review: Rayleigh-Ritz Approximation

Example: 1-D axially loaded bar

$$\Pi = \int_0^L \frac{1}{2} EA \left(\frac{dv}{dx}\right)^2 dx - \int_0^L q(x)v(x)dx - Pv(L)$$



Approximation: $v(x) = \sum_{i=1}^{N} a_i f_i(x)$ $\qquad f_i(x)$ - Basis functions (any complete set of interpolation functions)

$f_i(x) = x^{i-1}$ is an example

1. Satisfy Boundary Conditions: $v(0) = \sum_{i=1}^{N} a_i f_i(0) = 0$

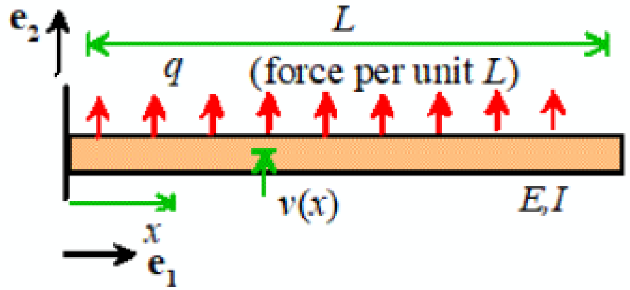
2. Eliminate some subset of a_i

3. Calculate PE $\Pi = \int_0^L \frac{1}{2} EA \left(\sum_{i=2}^{N} a_i \frac{df_i}{dx}\right)^2 dx - \int_0^L \sum_{i=2}^{N} a_i f_i(x) q(x) dx - P\sum_{i=2}^{N} a_i f_i(L)$

4. Minimize $\frac{\partial \Pi}{\partial a_i} = 0$

4. Solve for remaining a_i

Example: Use the Rayleigh-Ritz method to estimate the deflection of the beam



Follow general procedure

① Approximation $v = \sum_{i=1}^{N} a_i x^{(i-1)}$

② Boundary condition $v = 0$ @ $x = 0 \Rightarrow a_1 = 0$
$dv/dx = 0$ @ $x = 0 \Rightarrow a_2 = 0$

③ Find $\Pi = \int_0^L \frac{1}{2} E I \left(\frac{d^2 v}{dv^2}\right)^2 dx - \int_0^L q v \, dx$

④ Minimize : $\partial \Pi / \partial a_i = 0$ for $i = 3, 4 \ldots N$
Solve : subst back

Implement as general symbolic Matlab

```
clear all
syms a x n_terms v PI EE II L q
figure
for n_terms=3:6 % Try solutions between 3 and 6 terms
    clear eq
    a = sym('A',[n_terms,1]); % Create a vector containing the unknown coefficients
    v = 0; % Define the approximation
    for i=1:n_terms
        v=v+a(i)*x^(i-1);  %Adds each term to the approximation one at a time
    end
    n_constraints = 2;  % Enforce the constraints - v and dv/dx=0 at x=0
    eq(1) = subs(v,x,0)==0;
    eq(2) = subs(diff(v,x),x,0)==0;
    asol = struct2cell(solve(eq,a(1:n_constraints))); % Eliminate two of the unknowns
    for i=1:n_constraints
        v = subs(v,a(i),asol{i});
    end
    PI = int(EE*II/2*diff(v,x,2)^2 - q*v,x,[0,L]); % Potential energy
    for i=n_constraints+1:n_terms
        eq(i)=diff(PI,a(i))==0;  % Differentiate wrt all the remaining coefficients
    end
    if (n_terms>n_constraints+1) % Solve for the remaining unknowns
      asol = struct2cell(solve(eq(n_constraints+1:n_terms),a(n_constraints+1:n_terms)));
      for i=n_constraints+1:n_terms
       v = subs(v,a(i),asol{i-n_constraints});
      end
    else
        asol = solve(eq(n_constraints+1),a(n_constraints+1));
        v = subs(v,a(n_constraints+1),asol);
    end
    v % Display the results and plot it
    string = [num2str(n_terms) ' Terms'];
    fplot(subs(v,[EE,II,L,q],[1,1,1,1]),[0,1],'DisplayName',string)
    hold on
end
```

v =

$$\frac{L^2\,q\,x^2}{12\,\mathrm{EE}\,\mathrm{II}}$$ 3 terms

v =

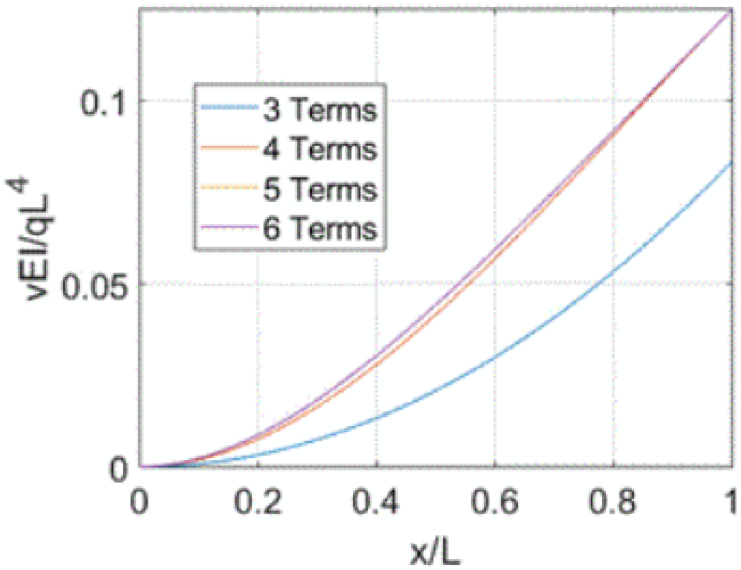
$$\frac{5\,L^2\,q\,x^2}{24\,\mathrm{EE}\,\mathrm{II}} - \frac{L\,q\,x^3}{12\,\mathrm{EE}\,\mathrm{II}}$$ 4 terms

v =

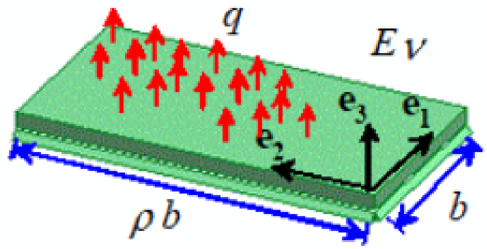
$$\frac{q\,x^4}{24\,\mathrm{EE}\,\mathrm{II}} - \frac{L\,q\,x^3}{6\,\mathrm{EE}\,\mathrm{II}} + \frac{L^2\,q\,x^2}{4\,\mathrm{EE}\,\mathrm{II}}$$ 5 terms

v =

$$\frac{q\,x^4}{24\,\mathrm{EE}\,\mathrm{II}} - \frac{L\,q\,x^3}{6\,\mathrm{EE}\,\mathrm{II}} + \frac{L^2\,q\,x^2}{4\,\mathrm{EE}\,\mathrm{II}}$$ 6 terms



Example: A flat rectangular plate with thickness h is subjected to uniform pressure. Its potential energy is



$$\Pi = \frac{Eh^3}{12(1-\nu)} \int_A \frac{1}{2}\left[\left(\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} \right)^2 - 2(1-\nu)\left(\frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - \left(\frac{\partial^2 v}{\partial x_1 \partial x_2} \right)^2 \right) \right] dA - \int_A q(x_1,x_2)v(x_1,x_2)\,dA$$

Use the approximation $v = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} \sin[\pi(2i-1)x/b]\sin[\pi(2j-1)y/\rho b]$ to estimate the deflection

Follow recipe:

Boundary conditions: $v=0$ on $x=0$, $x=b$
$v=0$ on $y=0$, $y=b$

Approximation satisfies this automatically

Find Π ; Minimize $\frac{\partial \Pi}{\partial A_{ij}} = 0$ for each A_{ij}

Solve; subst back, plot ; use MATLAB

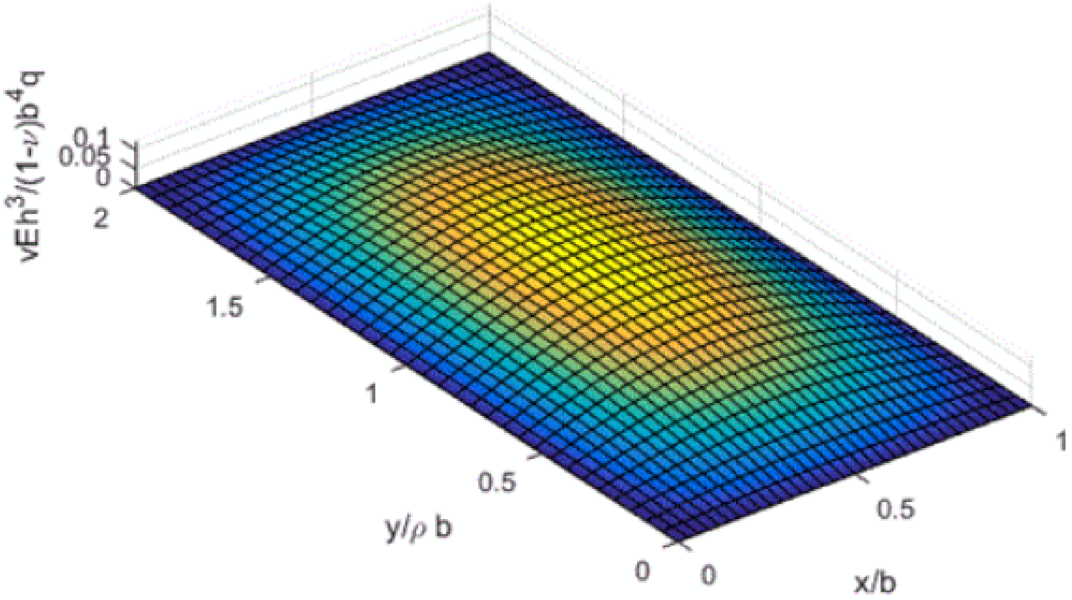
```
syms b rho h x y v PI h EE nu q U
n_terms = 2;
rho = 2;
a = sym('A',[n_terms,n_terms]);
v = 0;
for i=1:n_terms
    for j=1:n_terms
       v=v+a(i,j)*sin((2*i-1)*pi*x/b)*sin((2*j-1)*pi*y/(rho*b));
    end
end
U = simplify(EE*h^3/(24*(1-nu))*( (diff(v,x,2)+diff(v,y,2))^2 ...
    - 2*(1-nu)*( diff(v,x,2)*diff(v,y,2) - diff(diff(v,x),y)^2 ) ));
PI = int(int(U - q*v,x,[0,b]),y,[0,rho*b]);
count = 1;
for i = 1:n_terms
    for j = 1:n_terms
        eq(count) = diff(PI,a(i,j))==0;
        unknowns(count) = a(i,j);
        count = count + 1;
    end
end
asol = struct2cell(solve(eq,unknowns));

 for i=1:n_terms*n_terms
     v = subs(v,unknowns(i),asol{i});
 end
 simplify(v)
 fsurf(subs(v,[EE,nu,h,b,q],[1,0,1,1,1]),[0 1 0 rho])
 daspect([1 1 1])
```

ans =

$$\frac{4096\, b^4 q \sin\left(\frac{\pi x}{b}\right) \sin\left(\frac{\pi y}{2 b}\right) (v-1) \left(-925444 \sin\left(\frac{\pi x}{b}\right)^2 \sigma_1 + 1720758 \sin\left(\frac{\pi x}{b}\right)^2 + 9010758\, \sigma_1 - 49693617\right)}{1405518075\, \mathrm{EE}\, h^3 \pi^6}$$

where

$$\sigma_1 = \sin\left(\frac{\pi y}{2 b}\right)^2$$



Results: method works with $n = 2$
very slow with $n = 3$

$n > 3$: takes forever

Rayleigh-Ritz with symbolic solution is too slow: need a better method

Finite element method for linear elastic solid is a very efficient implementation of R-R

[Can be genealized to nonlinear problems]

10 Implementing FEA for linear elastic solids

For simplicity, assume :

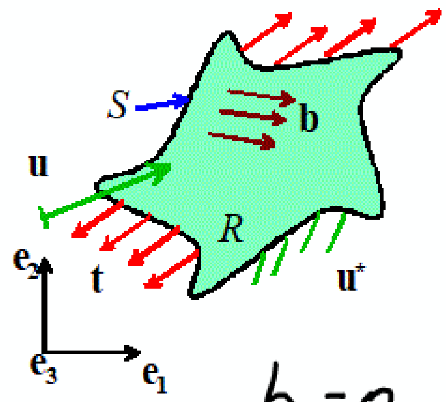
(1) Isotropic elastic material

(2) Small deformation

(3) neglect body forces, $\Delta T = 0$

(4) BCs: $\underline{u} = \underline{u}^*$ on S_1

$\underline{n}\,\sigma = \underline{t}$ on S_2



$\underline{b} = \underline{0}$

Procedure: (1) Approximate $\underline{u}$ (FE mesh)

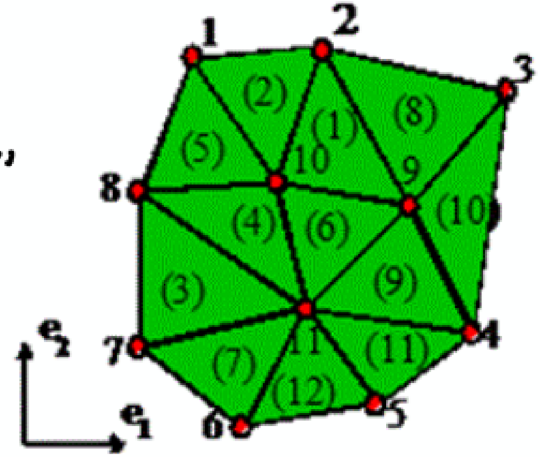
(2) Find Π

(3) Minimize Π

(4) Post-processing

To simplify focus on 2D plane strain problems

10.1 Finite element interpolation



Consider simplest element : "constant strain" triangles : poor elements in practice

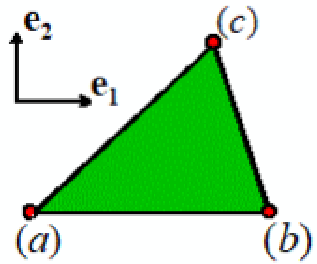
Divide solid into triangles

Let $\underline{x}^a$ denote coords of a^{th} vertex $a = 1, 2 \ldots N$

$\underline{u}^a$ " unknown displacement of a^{th} vertex

Use a linear interpolation to find $\underline{u}$ in each triangle

$$\underline{u} = N^a(\underline{x})\,\underline{u}^a + N^b(\underline{x})\,\underline{u}^b + N^c(\underline{x})\,\underline{u}^c$$



N^a, N^b, N^c are "Interpolation functions"

Interpolation functions are

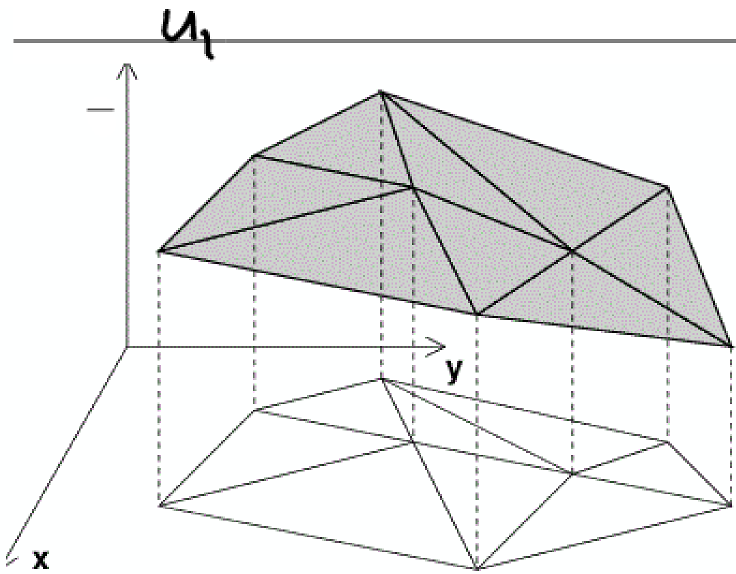
$$N^a(\underline{x}) : \frac{(x_2 - x_2^b)(x_1^c - x_1^b) - (x_1 - x_1^b)(x_2^c - x_2^b)}{(x_2^a - x_2^b)(x_1^c - x_1^b) - (x_1^a - x_1^b)(x_2^c - x_2^b)}$$

For N^b, N^c use cyclic permutation of a, b, c

$$N^a(x_1,x_2) = \frac{\left(x_2 - x_2^{(b)}\right)\left(x_1^{(c)} - x_1^{(b)}\right) - \left(x_1 - x_1^{(b)}\right)\left(x_2^{(c)} - x_2^{(b)}\right)}{\left(x_2^{(a)} - x_2^{(b)}\right)\left(x_1^{(c)} - x_1^{(b)}\right) - \left(x_1^{(a)} - x_1^{(b)}\right)\left(x_2^{(c)} - x_2^{(b)}\right)}$$

$$N^b(x_1,x_2) = \frac{\left(x_2 - x_2^{(c)}\right)\left(x_1^{(a)} - x_1^{(c)}\right) - \left(x_1 - x_1^{(c)}\right)\left(x_2^{(a)} - x_2^{(c)}\right)}{\left(x_2^{(b)} - x_2^{(c)}\right)\left(x_1^{(a)} - x_1^{(c)}\right) - \left(x_1^{(b)} - x_1^{(c)}\right)\left(x_2^{(a)} - x_2^{(c)}\right)}$$

$$N^c(x_1,x_2) = \frac{\left(x_2 - x_2^{(a)}\right)\left(x_1^{(b)} - x_1^{(a)}\right) - \left(x_1 - x_1^{(a)}\right)\left(x_2^{(b)} - x_2^{(a)}\right)}{\left(x_2^{(c)} - x_2^{(a)}\right)\left(x_1^{(b)} - x_1^{(a)}\right) - \left(x_1^{(c)} - x_1^{(a)}\right)\left(x_2^{(b)} - x_2^{(a)}\right)}$$



10.2 Calculating strains

- Plane strain $\Rightarrow$ $\varepsilon_{11} = \dfrac{\partial u_1}{\partial x_1}$ $\quad \varepsilon_{22} = \dfrac{\partial u_2}{\partial x_2}$ $\quad 2\varepsilon_{12} = \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}$

(all others zero)

Differentiate $\underset{\sim}{u}$ eg $\varepsilon_{11} = \dfrac{\partial N^a}{\partial x_1} u_1^a + \dfrac{\partial N^b}{\partial x_1} u_1^b + \dfrac{\partial N^c}{\partial x_1} u_1^c$

Express result in matrix form

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \underbrace{\begin{bmatrix} \partial N^a/\partial x_1 & 0 & \partial N^b/\partial x_1 & 0 & \partial N^c/\partial x_1 & 0 \\ 0 & \partial N^a/\partial x_2 & 0 & \partial N^b/\partial x_2 & 0 & \cdots \\ \dfrac{\partial N^a}{\partial x_2} & \dfrac{\partial N^a}{\partial x_1} & \dfrac{\partial N^b}{\partial x_2} & \dfrac{\partial N^b}{\partial x} & \dfrac{\partial N^c}{\partial x_2} & \dfrac{\partial N^c}{\partial x_1} \end{bmatrix}}_{[B]} \begin{bmatrix} u_1^a \\ u_2^a \\ u_1^b \\ u_2^b \\ u_1^c \\ u_2^c \end{bmatrix} \Leftarrow \underset{\sim}{u}^{el}$$

"Element displacement vector"

10.3 Strain energy density

Recall $\bar{U} = \frac{1}{2}\,\underline{\varepsilon} \cdot \underline{\sigma}$ $\qquad \underline{\sigma} = [\sigma_{11}\ \ \sigma_{22}\ \ \sigma_{12}]$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \underbrace{\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix}}_{[D]}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}$$

Hence $\bar{U} = \frac{1}{2}\,\underline{\varepsilon} \cdot \left([D]\,\underline{\varepsilon} \right)$ $\qquad \underline{\varepsilon} = [B]\,\underline{u}^{el}$

Hence $\bar{U} = \frac{1}{2}\left([B]\,\underline{u}^{el} \right) \cdot \left([D]\,[B]\,\underline{u}^{el} \right)$

$$= \frac{1}{2}\,\underline{u}^{el} \cdot \underbrace{\left\{ [B]^T [D]\,[B] \right\}}_{\text{6x6 matrix (constant for linear triangles)}} \underline{u}^{el}$$

10.4 Strain energy in one element

$$\bar{W}^{el} = \int_{A_{el}} \bar{U} \, dA \qquad - \text{Here } \bar{U} \text{ is constant}$$

$$= \frac{1}{2} \underset{\sim}{u}^{el} \cdot \underbrace{\left(A_{el} [B]^T [D][B] \right)}_{6\times 6 \text{ matrix } [k^{el}]} \underset{\sim}{u}^{el}$$

"Element stiffness matrix"