

EN2210: Continuum Mechanics

Homework 2: Kinematics Due Wed September 26, 2012

- 1. To analyze the deformation of a conical membrane, it is proposed to use a two-dimensional conical-polar coordinate system (s, θ) illustrated in the figure. s denotes the distance of a point on the cone from its apex, and θ is the angle subtended by a radial line and the **i** direction.
 - (a) Find the coordinate transformation from $\{x_1, x_2, x_3\}$ to s, θ and the inverse



(b) Find formulas for the three basis vectors

$$\mathbf{e}_{s} = \frac{\partial \mathbf{x} / \partial s}{\left|\partial \mathbf{x} / \partial s\right|}, \ \mathbf{e}_{\theta} = \frac{\partial \mathbf{x} / \partial \theta}{\left|\partial \mathbf{x} / \partial \theta\right|} \quad \mathbf{e}_{n} = \mathbf{e}_{s} \times \mathbf{e}_{\theta} \text{ in the } \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$$
basis.

(c) Hence, determine expressions for

in terms of $\{\mathbf{e}_s, \mathbf{e}_{\theta}, \mathbf{e}_n\}$

$$\frac{\partial \mathbf{e}_s}{\partial \theta} \quad \frac{\partial \mathbf{e}_{\theta}}{\partial \theta} \quad \frac{\partial \mathbf{e}_n}{\partial \theta}$$

(d) Let $d\mathbf{r} = ds\mathbf{e}_r + s\sin\alpha d\theta\mathbf{e}_{\theta}$ be an infinitesimal vector that lies in the surface of the cone. Find formulas for $ds, d\theta$ in terms of $d\mathbf{r}$ and other relevant variables.

(e) Let $\phi(s,\theta)$ be a scalar valued function defined on the surface of the cone. The *surface gradient* of $\nabla_s \phi$ is defined so that $[\nabla_s \phi] \cdot d\mathbf{r} = d\phi$ for all infinitesimal vectors that lie in the surface of the cone. Show that the surface gradient operator is

$$\nabla_s \equiv \left(\frac{\partial}{\partial s}\mathbf{e}_r + \frac{1}{s\sin\alpha}\frac{\partial}{\partial\theta}\mathbf{e}_\theta\right)$$

(f) The curvature tensor κ of a surface is defined so that $\kappa \cdot d\mathbf{r} = d\mathbf{n}$ gives the difference in normal to the surface \mathbf{n} at two points on the surface separated by an infinitesimal vector $d\mathbf{r}$. Use the solutions to (c) and (d) to determine the components of κ in $\{\mathbf{e}_s, \mathbf{e}_{\theta}, \mathbf{e}_n\}$

2. To track the deformation in a slowly moving glacier, three survey stations are installed in the shape of an equilateral triangle, spaced 100m apart, as shown in the picture. After a suitable period of time, the spacing between the three stations is measured again, and found to be 90m, 110m and 120m, as shown in the figure. Assuming that the deformation of the glacier is homogeneous over the region spanned by the survey stations, please compute the components of the Lagrange strain tensor associated with this deformation, expressing your answer as components in the basis shown.



3. A spherical shell (see the figure) is made from an incompressible material. In its undeformed state, the inner and outer radii of the shell are A, B. After deformation, the new values are a, b. The deformation in the shell can be described (in Cartesian components) by the equation

$$y_i = r \frac{x_i}{R}$$
 $r = \left(R^3 + a^3 - A^3\right)^{1/3}$ $R = \sqrt{x_k x_k}$

- (a) Calculate the components of the deformation gradient tensor
- P_a, a e₁ b e₂
- (b) Verify that the deformation is volume preserving
- (c) Find the deformed length of an infinitesimal radial line that has initial length l_0 , expressed as a function of *R*
- (d) Find the deformed length of an infinitesimal circumferential line that has initial length l_0 , expressed as a function of R
- (e) Using the results of (c) and (d), write down the principal stretches for the deformation.
- (f) Find the *inverse* of the deformation gradient, expressed as a function of y_i . You can do this by inspection, by inverting (a) (not recommended!), or by working out a formula that enables you to calculate x_i in terms of y_i and $r = \sqrt{y_i y_i}$ and differentiating the result. The first is quickest!

4. Suppose that the spherical shell described in Problem 3 is continuously expanding (visualize a balloon being inflated). The rate of expansion can be characterized by the velocity $v_a = da/dt$ of the surface that lies at R=A in the undeformed cylinder.

- (a) Calculate the velocity field $v_i = dy_i / dt$ in the sphere as a function of x_i
- (b) Calculate the velocity field as a function of y_i
- (c) Calculate the time derivative of the deformation gradient tensor calculated in 2(a)
- (d) Calculate the components of the velocity gradient $L_{ij} = \frac{\partial v_i}{\partial y_i}$ by differentiating the result of (b)
- (e) Calculate the components of the velocity gradient using the results of (c) and 3(f)
- (f) Calculate the stretch rate tensor D_{ij} . Verify that the result represents a volume preserving stretch rate field.

5. Repeat Problem 3(a), 3(f) and 4(b), 4(d), but this time solve the problem using spherical-polar coordinates, using the various formulas for vector and tensor operations given in the notes. In this case, you may assume that a point with position $\mathbf{x} = R\mathbf{e}_R$ in the undeformed solid has position vector

$$\mathbf{y} = \left(R^3 + a^3 - A^3\right)^{1/3} \mathbf{e}_R$$

after deformation.



6. An initially straight beam is bent into a circle with radius R as shown in the figure. Material fibers that are perpendicular to the axis of the undeformed beam are assumed to remain perpendicular to the axis after deformation, and the beam's thickness and the length of its axis are assumed to be unchanged. Under these conditions the deformation can be described as

$$y_1 = (R - x_2)\sin(x_1/R)$$
 $y_2 = R - (R - x_2)\cos(x_1/R)$

where, as usual \mathbf{x} is the position of a material particle in the undeformed beam, and \mathbf{y} is the position of the same particle after deformation.

- (a) Calculate the deformation gradient field in the beam, expressing your answer as a function of x_1, x_2 , and as components in the basis $\{e_1, e_2, e_3\}$ shown.
- (b) Calculate the Lagrange strain field in the beam.
- (c) Calculate the infinitesimal strain field in the beam.
- (d) Compare the values of Lagrange strain and infinitesimal strain for two points that lie at $(x_1 = 0, x_2 = h)$ and $(x_1 = L, x_2 = 0)$. Explain briefly the physical origin of the difference

between the two strain measures at each point. Recommend maximum allowable values of h/R and L/R for use of the infinitesimal strain measure in modeling beam deflections.

- (e) Calculate the deformed length of an infinitesimal material fiber that has length l_0 and orientation \mathbf{e}_1 in the undeformed beam. Express your answer as a function of x_2 .
- (f) Calculate the change in length of an infinitesimal material fiber that has length l_0 and orientation \mathbf{e}_2 in the undeformed beam.
- (g) Show that the two material fibers described in (3) and (f) remain mutually perpendicular after deformation. Is this true for *all* material fibers that are mutually perpendicular in the undeformed solid?
- (h) Find the components in the basis $\{e_1, e_2, e_3\}$ of the Left and Right stretch tensors U and V as well as the rotation tensor **R** for this deformation. You should be able to write down U and **R** by inspection, without needing to wade through the laborious general process. The results can then be used to calculate V.
- (i) Find the principal directions of \mathbf{U} as well as the principal stretches. You should be able to write these down without doing any tedious calculations.
- (j) Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be a basis in which \mathbf{m}_1 is parallel to the axis of the deformed beam, as shown in the figure. Write down the components of each of the unit vectors \mathbf{m}_i in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Hence, compute the transformation matrix $Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$ that is used to transform tensor components from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{m}_1, \mathbf{m}_2\}$
- (k) Find the components of the deformation gradient tensor, Lagrange strain tensor, as well as U V and R in the basis $\{m_1, m_2, m_3\}$. It is best to do these with a symbolic manipulation program.
- (1) find the principal directions of **V** expressed as components in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. Again, you should be able to simply write down this result.





7. A sheet of material is subjected to a two dimensional homogeneous deformation of the form

$$y_1 = A_{11}x_1 + A_{12}x_2$$
 $y_2 = A_{21}x_1 + A_{22}x_2$

where A_{ij} are constants. Suppose that a circle of unit radius is

drawn on the undeformed sheet. This circle is distorted to a smooth curve on the deformed sheet. Show that the distorted circle is an ellipse, with semi-axes that are parallel to the



principal directions of the left stretch tensor \mathbf{V} , and that the lengths of the semi-axes of the ellipse are equal to the principal stretches for the deformation. There are many different ways to approach this calculation – some are very involved. The simplest way is probably to use the polar decomposition $\mathbf{A} = \mathbf{V} \cdot \mathbf{R}$.

8. The center of mass and the mass moment of inertia tensor in the reference and deformed configurations of a solid are (by definition)

$$r_{i}^{c0} = \frac{1}{M} \int_{V_{0}} x_{i} \rho_{0} dV_{0} \qquad I_{ij}^{c0} = \int_{V_{0}} \left(x_{i} - r_{i}^{c0} \right) \left(x_{j} - r_{j}^{c0} \right) \rho_{0} dV_{0}$$
$$r_{i}^{c} = \frac{1}{M} \int_{V} y_{i} \rho dV \qquad I_{ij}^{c} = \int_{V} \left(y_{i} - r_{i}^{c} \right) \left(y_{j} - r_{j}^{c} \right) \rho dV$$

where ρ_0 , ρ are the mass density of the solid in the reference and deformed configurations, **x**, **y** are the positions of material particles in the reference and deformed configurations, and *M* is the total mass.

Suppose that a solid is subjected to a homogeneous deformation

$$y_i = A_{ik} x_k + c_i$$

where A_{ii} and c_i are constants.

- (a) Find formulas for r_{ic} , I_{ij}^C in terms of r_i^{c0} , I_{ij}^{c0} , A_{ij} and c_i .
- (b) Suppose that A_{ij} is a rigid rotation (this means $A_{ik}A_{jk} = A_{ki}A_{kj} = \delta_{ij}$. Use the solution to (a) to show that the time derivative of I_{ij} can be expressed as

$$\frac{dI_{ij}^c}{dt} = W_{ik}I_{kj}^c - I_{ik}^cW_{kj}$$

where $W_{ik} = \frac{dA_{ik}}{dt}A_{jk}$ is the spin tensor.

(c) Suppose that a rigid body rotates with angular velocity ω_k and therefore has angular momentum

$$h_i = I_{ij}^c \omega_j$$

Use (b) to show that the time derivative of the angular momentum is

$$\frac{dh_i}{dt} = I_{ij} \frac{d\omega_j}{dt} + \epsilon_{ijk} \,\omega_j I_{kl} \omega_l$$



9. The figure shows a design for a high-speed moving walkway (see http://www.jfe-steel.co.jp/archives/en/nkk_giho/84/pdf/84_10.pdf for a detailed description of this general type of design, or http://www.youtube.com/watch?v=uwHer1RrYg8 for a movie of such a walkway in action). A passenger standing on the walkway passes through five regions:

- (i) between A and B she moves at constant speed v_0 ;
- (ii) between B and C she accelerates (with an acceleration to be specified below);
- (iii) between C and D she moves with constant (high) speed v_1 ; and
- (iv) between D and E she decelerates
- (v) between E and F she travels at speed v_0 again.

In this problem we will just focus on portion (ii) of the motion -i.e. between B and C.

- (a) Suppose that the walkway is designed so that the velocity varies linearly with distance between B and C. Assume that a person walks with speed w relative to the moving walkway. Determine her acceleration as a function of distance y from B, and also as a function of time after passing the point B. Find a formula for the maximum value of the acceleration, and identify the point where it occurs.
- (b) Suppose the walkway is designed instead so that a person standing on the track has constant acceleration *a*. Calculate the required velocity distribution v(y) as a function of distance *y* from B, and determine the acceleration of the person walking along the accelerating walkway as a function of *y* and also a function of *t*.