School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 2: Polar and Curvilinear Coordinates, Kinematics
Due Wednesday September 28, 2016

1. The for the vector $v_i = \dfrac{x_i}{R^3}$ and tensor $S_{ij} = \dfrac{\delta_{ij}}{R^3} + \dfrac{x_i x_j}{R^5}$ $\quad R = \sqrt{x_k x_k}$, calculate:

a. Their components in spherical-polar coordinates
b. The gradient of **v** in spherical-polar coordinates
c. The divergence of **S** in spherical-polar coordinates

You can do this with the basis change formulas, of course, but there is a much easier way using physical insight – think about the direction of the vector with components x_i / R , and think also about what the tensor δ_{ij} must look like in polar coordinates.

2. 'Parabolic Coordinates' are used to simplify the solution of PDEs for solids with parabolic boundaries. They specify the position of a point using three parametric coordinates (u, v, θ) as

$$\mathbf{r} = uv(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + \frac{1}{2}(u^2 - v^2)\mathbf{k}$$

(a) Find the components of normalized basis vectors for this coordinate system

$$\mathbf{e}_u = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial u}\right|}\frac{\partial \mathbf{r}}{\partial u} \qquad \mathbf{e}_v = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial v}\right|}\frac{\partial \mathbf{r}}{\partial u} \qquad \mathbf{e}_\theta = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|}\frac{\partial \mathbf{r}}{\partial \theta}$$

Show that they are orthogonal; and calculate their derivatives with respect to (u, v, θ) (express your answer in $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_\theta\}$ coordinates.

(b) Find an expression for the gradient operator in the (u, v, θ) system (express your answer in $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_\theta\}$ coordinates.

(c) Find the gradient and divergence of the vector field $\mathbf{w} = \dfrac{1}{u^2}\mathbf{e}_u$

3. Helical coordinates are used to reduce the governing equations for problems with helical symmetry (such as flow down a helical channel) to two dimensions. A number of different helical coordinate systems have been proposed. One example is defined by expressing position vector in terms of $(r, \theta, z) \equiv (\xi_1, \xi_2, \xi_3)$ as follows

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + (z + \frac{L}{2\pi}\theta)\mathbf{k}$$

Where L is the pitch of the helix.

(a) Calculate the covariant basis vectors $\mathbf{m}_i$

(b) Find expressions for the reciprocal basis vectors $\mathbf{m}^i$

(c) Calculate the covariant and contravariant components of the metric tensor $\mathbf{g}$. Check your answer by calculating $g_{ik}g^{kj}$

(d) Calculate the elements of Christoffel symbol (which satisfies $d\mathbf{m}_i = \Gamma_{ij}^{k}\mathbf{m}_k d\xi_j$)

(e) Calculate expressions for the covariant and contravariant components of the gradient of a scalar function ϕ

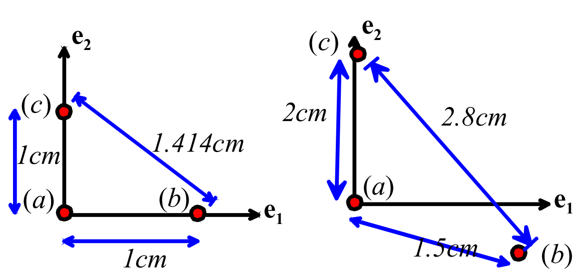
(f) Find expressions for the contravariant components of the velocity and acceleration of a particle, in terms of time derivatives of r, θ, z

(g) A steady flow down a helical channel must have the form $\mathbf{v} = v(r)\mathbf{m}_2$ (note that v does not have units of velocity because $\mathbf{m}_2$ is not a unit vector). Calculate the mixed components of the velocity gradient tensor $\mathbf{L} = \nabla\mathbf{v}$ such that $\mathbf{L} = L^{i}_{.j}\mathbf{m}_i \otimes \mathbf{m}^j$

4. Construct (i.e. find a displacement field) a homogeneous deformation that has the following properties:

- The volume of the solid is doubled
- A material fiber parallel to the $\mathbf{e}_1$ direction in the undeformed solid increases its length by a factor of $\sqrt{2}$ and is oriented parallel to the $\mathbf{e}_1 + \mathbf{e}_2$ direction in the deformed solid
- A material fiber parallel to the $\mathbf{e}_2$ direction in the undeformed is oriented parallel to the $-\mathbf{e}_1 + \mathbf{e}_2$ direction in the deformed solid.
- A material fiber parallel to the $\mathbf{e}_3$ direction in the undeformed solid preserves its length and orientation in the deformed solid

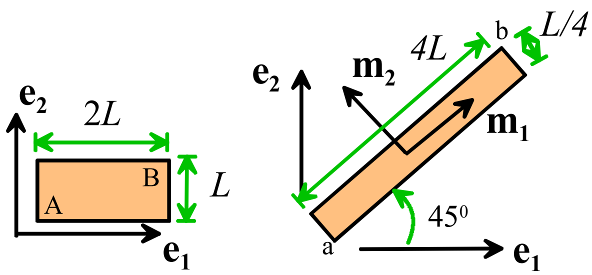
5. To measure the in-plane deformation of a sheet of metal during a forming process, your managers place three small hardness indentations on the sheet. Using a travelling microscope, they determine that the initial lengths of the sides of the triangle formed by the three indents are 1cm, 1cm, 1.414cm, as shown in the picture below. After deformation, the sides have lengths 1.5cm, 2.0cm and 2.8cm.



5.1 Calculate the components of the Lagrange strain tensor E_{11}, E_{22}, E_{12} in the basis shown.

5.2 Calculate the components of the Eulerian strain tensor $E_{11}^*,\ E_{22}^*,\ E_{12}^*$ in the basis shown.

6. The figure shows the reference and deformed configurations for a solid. The out-of-plane dimensions are unchanged. Points a and b are the positions of points A and B after deformation. Determine



6.1 The right stretch tensor **U**, expressed as components in $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. (A 2x2 matrix is sufficient). There is no need for lengthy calculations – you may write down the result by inspection.

6.2 The rotation tensor **R** in the polar decomposition of the deformation gradient **F**=**RU**=**VR**

6.3 The deformation gradient, expressed as components in $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$. Try to do this without using the basis-change formulas.

7. Show that the Lagrange strain **E**, the right Cauchy-Green deformation tensor **C** and the right stretch tensor **U** have the same principal directions (eigenvectors). Similarly, show that $\mathbf{E}^*, \mathbf{B}, \mathbf{V}$ have the same principal directions.