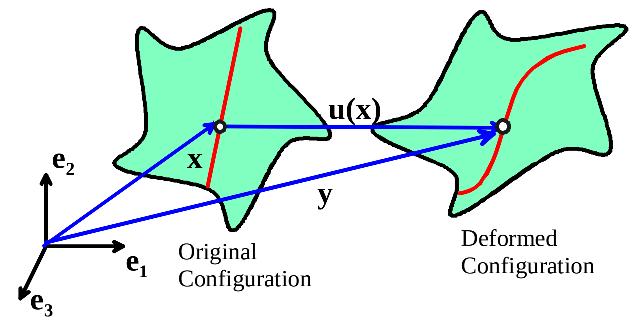
Kinematics

Deformation Mapping

$$y_i = x_i + u_i(x_1, x_2, x_3, t)$$



Eulerian/Lagrangian descriptions of motion

$$y_i = x_i + u_i(x_j, t) \qquad \left.\frac{\partial y_i}{\partial t}\right|_{x_i=\text{const}} = \frac{\partial u_i}{\partial t} = v_i(x_j, t)$$

$$y_i = x_i + u_i(y_j, t) \qquad \left.\frac{\partial y_i}{\partial t}\right|_{x_i=\text{const}} = v_i(y_j, t) \qquad \left.\frac{\partial^2 y_i}{\partial t^2}\right|_{x_i=\text{const}} = a_i(y_j, t)$$

$$\left(\delta_{ik} - \frac{\partial u_i}{\partial y_k}\right)\left.\frac{\partial y_k}{\partial t}\right|_{x_i=\text{const}} = \left.\frac{\partial u_i}{\partial t}\right|_{y_i=\text{const}} \qquad \left.\frac{\partial^2 y_i}{\partial t^2}\right|_{x_i=\text{const}} = a_i(y_j, t) = \left.\frac{\partial v_i}{\partial t}\right|_{y_i=\text{const}} + v_k(y_j, t)\frac{\partial v_i}{\partial y_k}$$

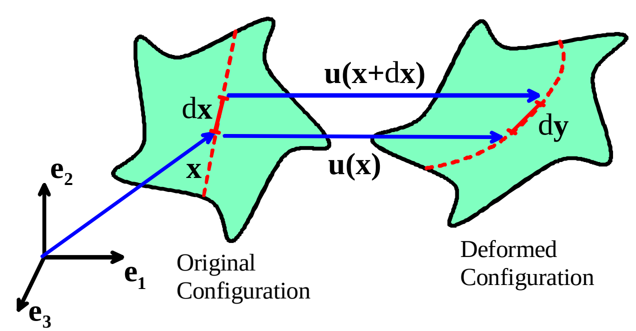
Deformation Gradient

$$\nabla \mathbf{y} = \nabla\left(\mathbf{x} + \mathbf{u}(\mathbf{x})\right) = \mathbf{F}$$

or $$\frac{\partial y_i}{\partial x_j} = \frac{\partial}{\partial x_j}(x_i + u_i) = \delta_{ij} + \frac{\partial u_i}{\partial x_j} = F_{ij}$$

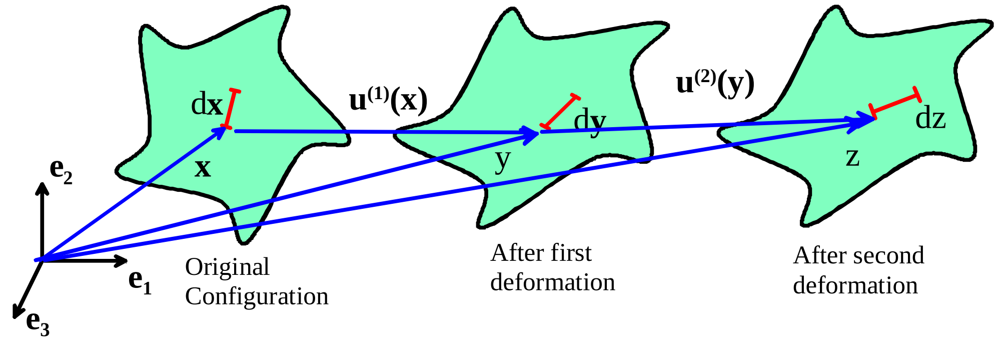
$$d\mathbf{y} = \mathbf{F} \cdot d\mathbf{x}$$

$$dy_i = F_{ik}\, dx_k$$



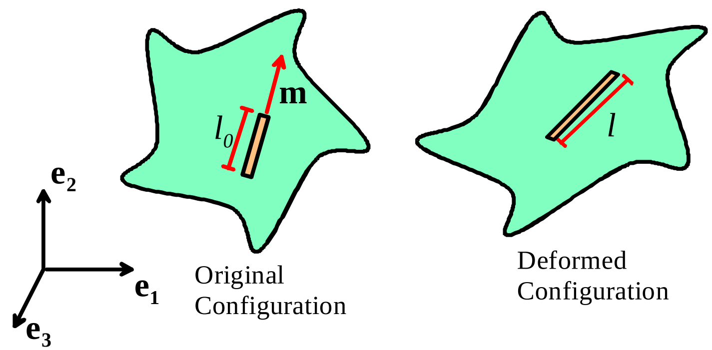
Kinematics

Sequence of deformations



$$d\mathbf{z} = \mathbf{F} \cdot d\mathbf{x} \quad \text{with} \quad \mathbf{F} = \mathbf{F}^{(2)} \cdot \mathbf{F}^{(1)} \quad \text{or} \quad dz_i = F_{ij} dx_j \qquad F_{ij} = F_{ik}^{(2)} F_{kj}^{(1)}$$

Lagrange Strain



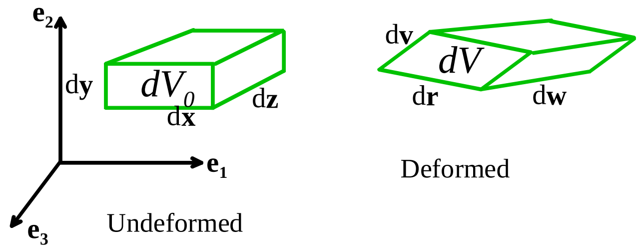
$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) \quad \text{or} \quad E_{ij} = \frac{1}{2}(F_{ki} F_{kj} - \delta_{ij})$$

$$\mathbf{m} \cdot \mathbf{E} \cdot \mathbf{m} = E_{ij} m_i m_j = \frac{l^2 - l_0{}^2}{2 l_0{}^2} = \frac{\delta l}{l_0} + \frac{(\delta l)^2}{2 l_0^2}$$

Kinematics

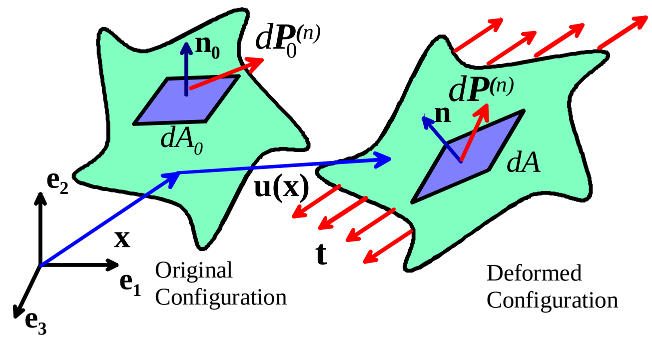
Volume Changes

$$\frac{dV}{dV_0} = \det(\mathbf{F}) = J$$



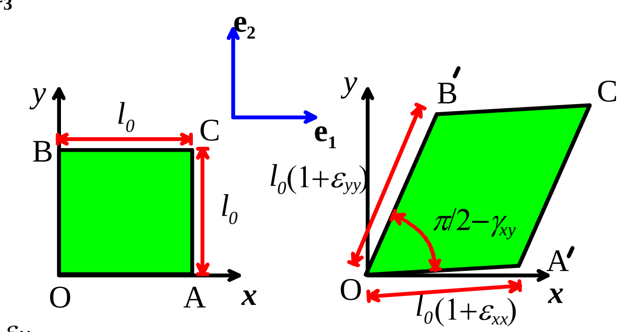
Area Elements

$$dA\mathbf{n} = J\mathbf{F}^{-T} \cdot dA_0\mathbf{n}_0 \qquad dAn_i = JF_{ki}^{-1} n_k^0 dA_0$$



Infinitesimal Strain

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{u}\nabla + (\mathbf{u}\nabla)^T\right) \quad \text{or} \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$



Approximates L-strain

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_i}\right) \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \approx \varepsilon_{ij}$$

Related to ‘Engineering Strains’

$$\varepsilon_{11} = \varepsilon_{xx}$$

$$\varepsilon_{22} = \varepsilon_{yy}$$

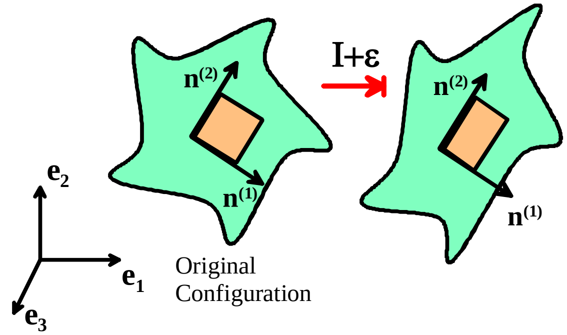
$$\varepsilon_{12} = \varepsilon_{21} = \gamma_{xy}/2 = \gamma_{yx}/2$$

Kinematics

Principal values/directions of Infinitesimal Strain

$$\boldsymbol{\varepsilon}\cdot\mathbf{n}^{(i)} = e_i\mathbf{n}^{(i)}$$

$$\text{or}\quad \varepsilon_{kl}n_l^{(i)} = e_i n_l^{(i)}$$

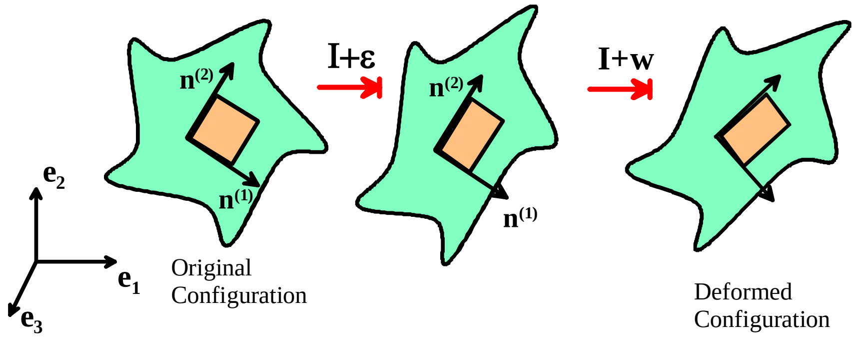


Infinitesimal rotation

$$\mathbf{w} = \frac{1}{2}\left(\mathbf{u}\nabla - (\mathbf{u}\nabla)^T\right) \quad \text{or} \quad w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

Decomposition of infinitesimal motion

$$\frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + w_{ij}$$



Kinematics

Left and Right stretch tensors, rotation tensor

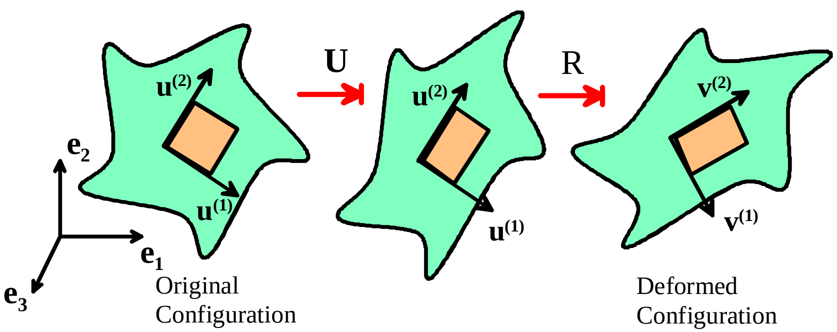
$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$$
$$\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$$

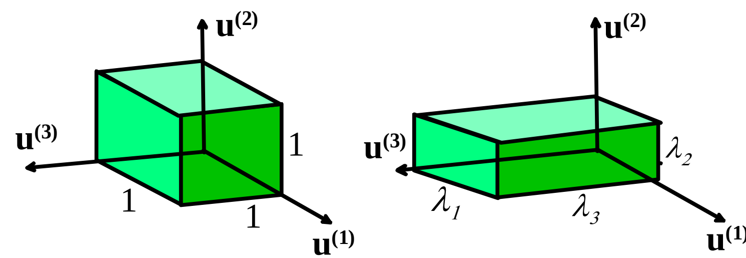
U,V symmetric, so

$$\mathbf{U} = \lambda_1 \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2 \mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \lambda_3 \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}$$

$$\mathbf{V} = \lambda_1 \mathbf{v}^{(1)} \otimes \mathbf{v}^{(1)} + \lambda_2 \mathbf{v}^{(2)} \otimes \mathbf{v}^{(2)} + \lambda_3 \mathbf{v}^{(3)} \otimes \mathbf{v}^{(3)}$$

λ_i principal stretches





Left and Right Cauchy-Green Tensors

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = \mathbf{U}^2$$
$$\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T = \mathbf{V}^2$$

Kinematics

Generalized strain measures

Lagrangian Nominal strain: $\sum_{i=1}^{3}(\lambda_i - 1)\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}$

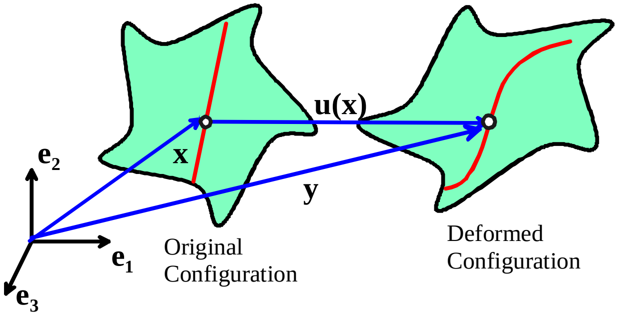
Lagrangian Logarithmic strain: $\sum_{i=1}^{3}\log(\lambda_i)\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}$

Eulerian Nominal strain: $\sum_{i=1}^{3}(\lambda_i - 1)\mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}$

Eulerian Logarithmic strain: $\sum_{i=1}^{3}\log(\lambda_i)\mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}$

Eulerian strain $\mathbf{E}^{*} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1})$ or $E_{ij}^{*} = \frac{1}{2}(\delta_{ij} - F_{ki}^{-1}F_{kj}^{-1})$

Kinematics



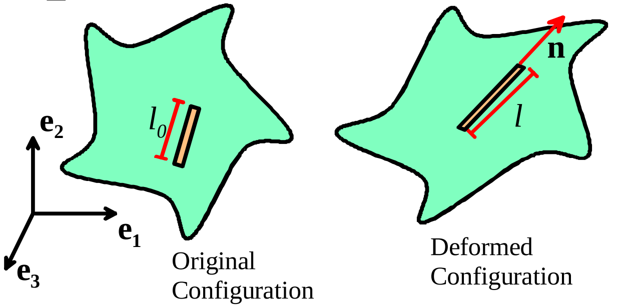
Velocity Gradient $$\mathbf{L} = \nabla_{\mathbf{y}} \mathbf{v} \equiv L_{ij} = \frac{\partial v_i}{\partial y_j}$$

$$dv_i = v_i(\mathbf{y} + d\mathbf{y}) - v_i(\mathbf{y}) = \frac{\partial v_i}{\partial y_j} dy_j$$

$$dv_i = \frac{d}{dt} dy_i = \frac{d}{dt}\left(F_{ij} dx_j\right) = \dot{F}_{ij} dx_j \qquad dv_i = \dot{F}_{ij} F_{jk}^{-1} dy_k \qquad \mathbf{L} = \frac{d\mathbf{F}}{dt} \mathbf{F}^{-1}$$

Stretch rate and spin tensors $$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) \qquad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)$$

$$\frac{1}{l}\frac{dl}{dt} = \mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n} = n_i D_{ij} n_j$$



Kinematics

Vorticity vector $\boldsymbol{\omega} = curl(\mathbf{v})$ $\omega_i = \in_{ijk} \dfrac{\partial v_k}{\partial y_j}$ $\boldsymbol{\omega} = 2dual(\mathbf{W})$ $\omega_i = -\in_{ijk} W_{jk}$

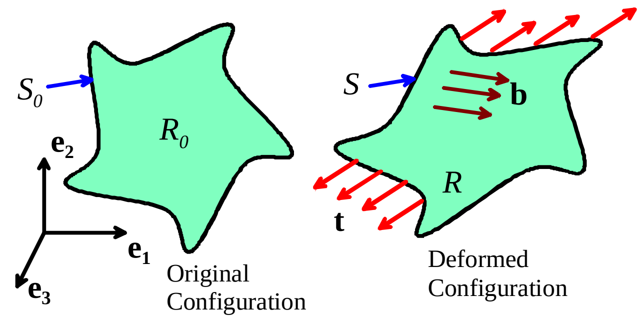
Spin-acceleration-vorticity relations

$$a_i = \left.\frac{\partial v_i}{\partial t}\right|_{x_k = const} = \left.\frac{\partial v_i}{\partial t}\right|_{y_k = const} + \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + 2W_{ik} v_k$$

$$a_i = \left.\frac{\partial v_i}{\partial t}\right|_{x_k = const} = \left.\frac{\partial v_i}{\partial t}\right|_{y_k = const} + \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + \in_{ijk} \omega_j v_k$$

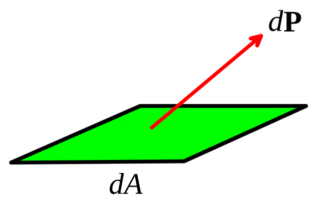
$$\in_{ijk} \frac{\partial a_k}{\partial y_j} = \left.\frac{\partial \omega_i}{\partial t}\right|_{\mathbf{x} = const} - D_{ij}\omega_j + \frac{\partial v_k}{\partial y_k}\omega_i$$

Kinetics



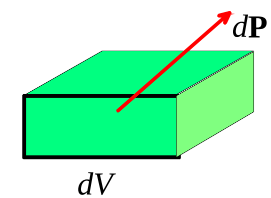
Surface traction

$$\mathbf{t} = \lim_{dA \to 0} \frac{d\mathbf{P}}{dA}$$



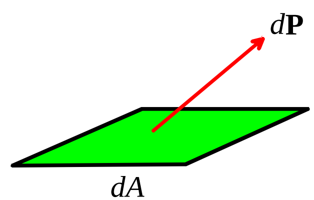
Body Force

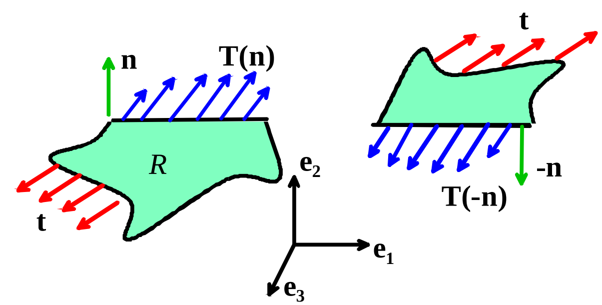
$$\rho \mathbf{b} = \lim_{dV \to 0} \frac{d\mathbf{P}}{dV}$$



Internal Traction

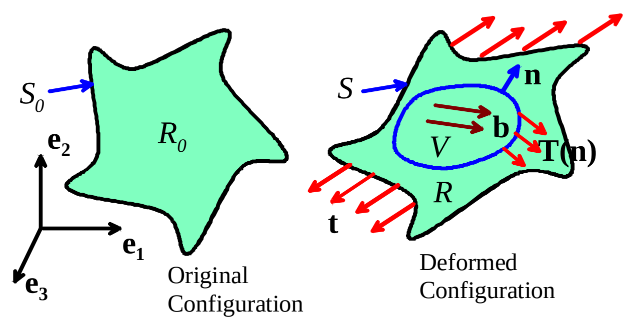
$$\mathbf{T}(\mathbf{n}) = \lim_{dA \to 0} \frac{d\mathbf{P}^{(\mathbf{n})}}{dA}$$





Resultant force on a volume

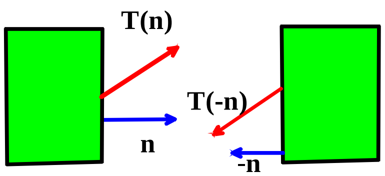
$$\mathbf{P} = \int_A \mathbf{T}(\mathbf{n})\, dA + \int_V \rho \mathbf{b}\, dV$$



Kinetics

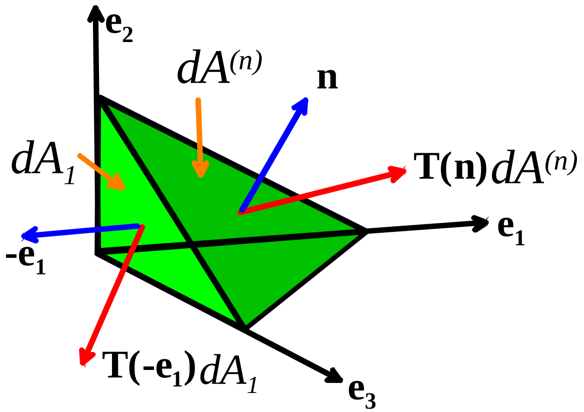
Restrictions on internal traction vector

Newton II $\mathbf{T}(-\mathbf{n}) = -\mathbf{T}(\mathbf{n})$

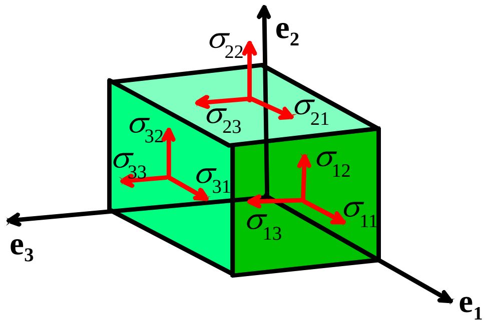


Newton II&III

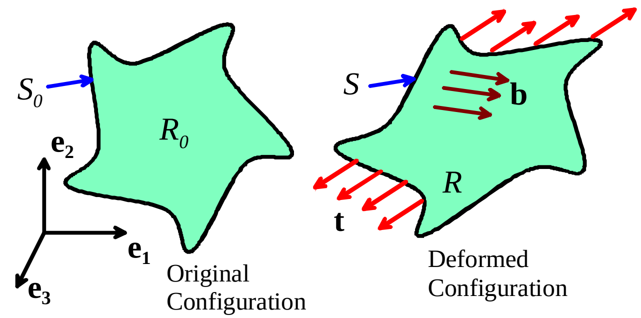
$$\mathbf{T}(\mathbf{n}) = \mathbf{n} \cdot \boldsymbol{\sigma} \quad \text{or} \quad T_i(\mathbf{n}) = n_j \sigma_{ji}$$



Cauchy Stress Tensor

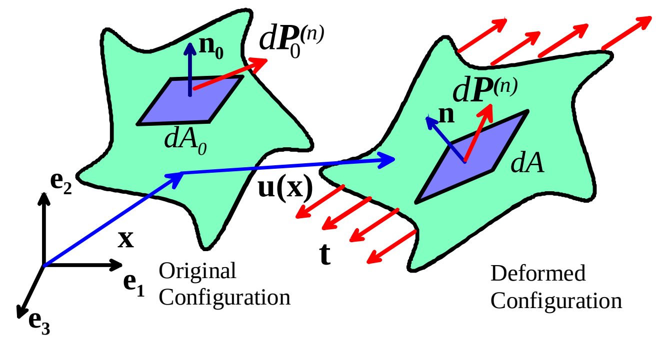


Other Stress Measures



$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u} \qquad F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$$

$$J = \det(\mathbf{F})$$



Kirchhoff

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \quad \tau_{ij} = J\sigma_{ij}$$

Nominal/ 1st Piola-Kirchhoff

$$\mathbf{S} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \qquad S_{ij} = JF_{ik}^{-1}\sigma_{kj}$$

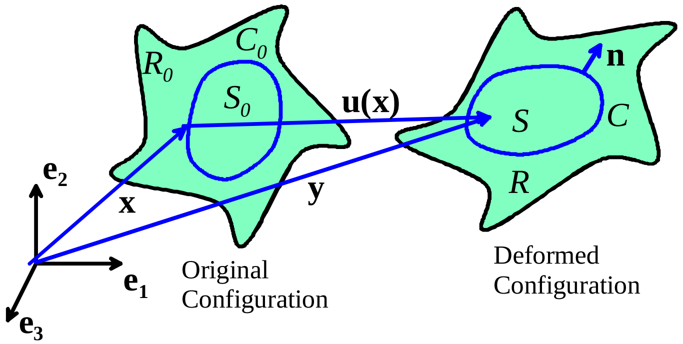
$$dP_j^{(\mathbf{n})} = dA_0 n_i^0 S_{ij}$$

Material/2nd Piola-Kirchhoff

$$\boldsymbol{\Sigma} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T} \quad \Sigma_{ij} = JF_{ik}^{-1}\sigma_{kl}F_{jl}^{-1}$$

$$F_{ij}dP_j^{(\mathbf{n}0)} = dP_i^{(\mathbf{n})} \qquad dP_i^{(\mathbf{n}0)} = dA_0 n_j^0 \Sigma_{ji}$$

Reynolds Transport Relation

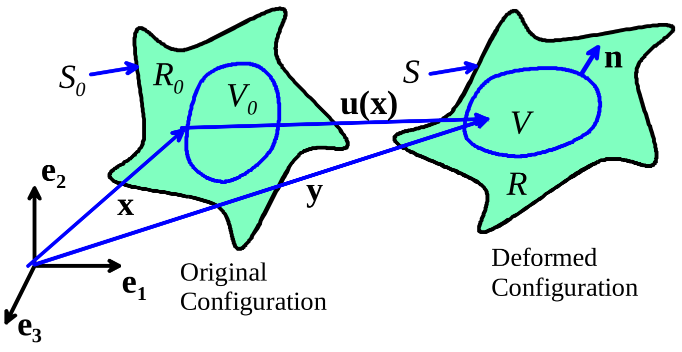


$$\frac{d}{dt}\int_V \phi dV = \int_V \left(\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \phi \frac{\partial v_i}{\partial y_j} \right) dV = \int_V \left(\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \phi v_i}{\partial y_j} \right) dV$$

Conservation Laws for Continua

Mass Conservation

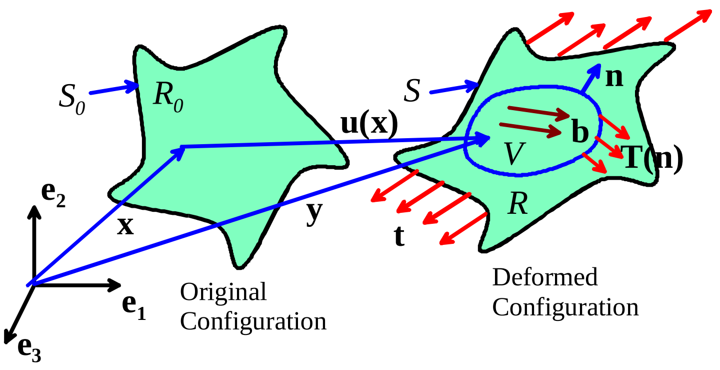
$$\int_V \left(\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho \frac{\partial v_i}{\partial y_i} \right) dV = 0 \Rightarrow \left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho \frac{\partial v_i}{\partial y_i} = 0$$



$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0 \qquad \left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \nabla_{\mathbf{y}} \cdot (\rho \mathbf{v}) = 0$$

Linear Momentum Conservation

$$\nabla_{\mathbf{y}} \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a} \quad \text{or} \quad \frac{\partial \sigma_{ij}}{\partial y_i} + \rho b_j = \rho a_j$$

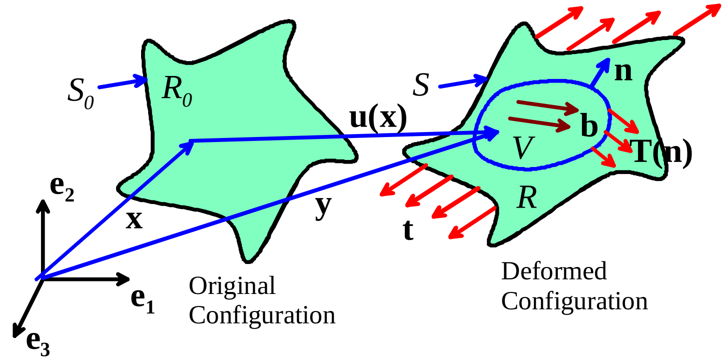


Angular Momentum Conservation

$$\sigma_{ij} = \sigma_{ji}$$

Work-Energy Relations

Rate of mechanical work done on a material volume



$$\dot{r} = \int_A T_i^{(\mathbf{n})} v_i dA + \int_V \rho b_i v_i dV = \int_V \sigma_{ij} D_{ij} dV + \frac{d}{dt}\left\{ \int_V \frac{1}{2} \rho v_i v_i dV \right\}$$

Conservation laws in terms of other stresses

$$\nabla \cdot \mathbf{S} + \rho_0 \mathbf{b} = \rho_0 \mathbf{a} \qquad \frac{S_{ij}}{\partial x_i} + \rho_0 b_j = \rho_0 a_j \qquad \nabla \cdot \left[\boldsymbol{\Sigma} \cdot \mathbf{F}^{\mathbf{T}} \right] + \rho_0 \mathbf{b} = \rho_0 \mathbf{a} \qquad \frac{\partial \left(\Sigma_{ik} F_{jk} \right)}{\partial x_i} + \rho_0 b_j = \rho_0 a_j$$

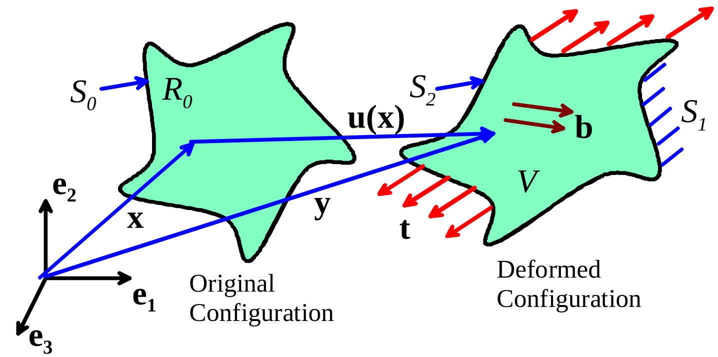
Mechanical work in terms of other stresses

$$\dot{r} = \int_A T_i^{(\mathbf{n})} v_i dA + \int_V \rho b_i v_i dV = \int_{V_0} S_{ij} \dot{F}_{ji} dV_0 + \frac{d}{dt}\left\{ \int_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0 \right\}$$

$$\dot{r} = \int_A T_i^{(\mathbf{n})} v_i dA + \int_V \rho b_i v_i dV = \int_{V_0} \Sigma_{ij} \dot{E}_{ij} dV_0 + \frac{d}{dt}\left\{ \int_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0 \right\}$$

Principle of Virtual Work (alternative statement of BLM)

$$\delta L_{ij} = \frac{\partial \delta v_i}{\partial y_j} \qquad \delta D_{ij} = \frac{1}{2}\left(\frac{\partial \delta v_i}{\partial y_j} + \frac{\partial \delta v_j}{\partial y_i} \right)$$

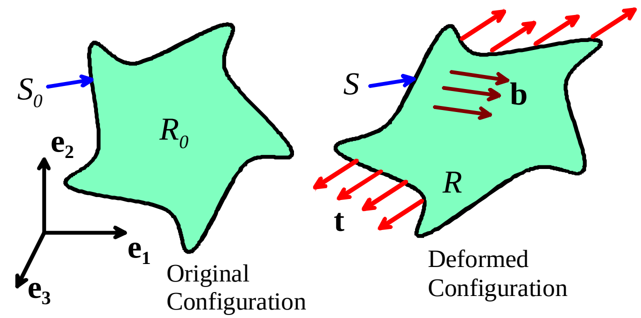


If $$\int_V \sigma_{ij}\delta D_{ij}\, dV + \int_V \rho \frac{dv_i}{dt}\delta v_i dV - \int_V \rho b_i \delta v_i dV - \int_{S_2} t_i \delta v_i dA = 0$$ for all δv_i

Then $$\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho \frac{dv_i}{dt}$$

$$n_i \sigma_{ij} = t_j \quad \text{on} \quad S_2$$

Thermodynamics



Temperature θ

Specific Internal Energy ε

Specific Helmholtz free energy $\psi = \varepsilon - \theta s$

Heat flux vector $\mathbf{q}$

External heat flux q

Specific entropy s

First Law of Thermodynamics
$$\frac{d}{dt}(\mathrm{E} + KE) = Q + W$$

$$\rho \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \sigma_{ij} D_{ij} - \frac{\partial q_i}{\partial y_i} + q$$

Second Law of Thermodynamics
$$\frac{dS}{dt} - \frac{d\mathrm{H}}{dt} \geq 0$$

$$\rho \frac{\partial s}{\partial t} + \frac{\partial (q_i / \theta)}{\partial y_i} - \frac{q}{\theta} \geq 0$$

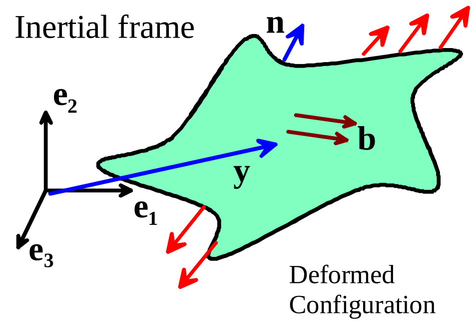
Dissipation Inequality
$$\sigma_{ij} D_{ij} - \frac{1}{\theta} q_i \frac{\partial \theta}{\partial y_i} - \rho \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0$$

Transformations under observer changes

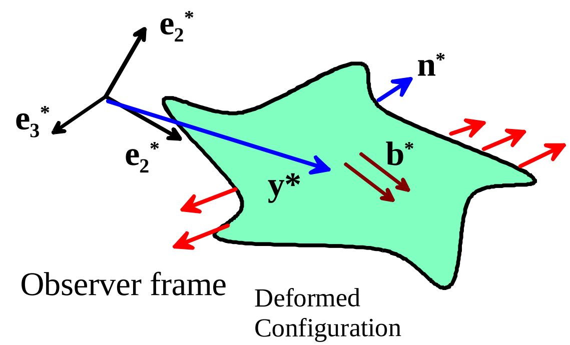
Transformation of space under
a change of observer

$$\mathbf{y}^* = \mathbf{y}_0^*(t) + \mathbf{Q}(t)(\mathbf{y} - \mathbf{y}_0)$$

$$\mathbf{\Omega} = \frac{d\mathbf{Q}}{dt}\mathbf{Q}^T$$



All physically measurable vectors can be regarded as connecting two points in the inertial frame



These must therefore transform like vectors connecting two points under a change of observer

$$\mathbf{b}^* = \mathbf{Q}\mathbf{b} \quad \mathbf{n}^* = \mathbf{Q}\mathbf{n} \quad \mathbf{v}^* = \mathbf{Q}\mathbf{v} \quad \mathbf{a}^* = \mathbf{Q}\mathbf{a}$$

Note that time derivatives in the observer's reference frame have to account for rotation of the reference frame

$$\mathbf{v}^* = \mathbf{Q}\mathbf{v} = \mathbf{Q}\frac{d\mathbf{y}}{dt} = \mathbf{Q}\frac{d}{dt}\mathbf{Q}^T(\mathbf{y}^* - \mathbf{y}_0^*(t)) = \frac{d\mathbf{y}^*}{dt} - \frac{d\mathbf{y}_0^*}{dt} - \mathbf{\Omega}(\mathbf{y}^* - \mathbf{y}_0^*(t))$$

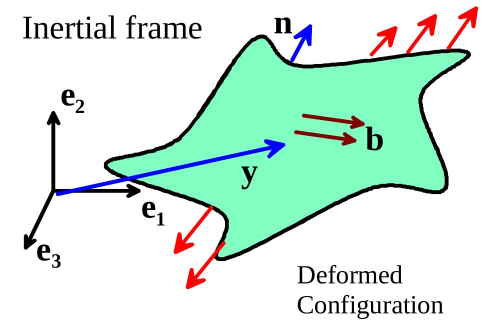
$$\mathbf{a}^* = \mathbf{Q}\mathbf{a} = \mathbf{Q}\frac{d^2\mathbf{y}}{dt^2} = \mathbf{Q}\frac{d^2}{dt^2}\mathbf{Q}^T(\mathbf{y}^* - \mathbf{y}_0^*(t)) = \frac{d^2\mathbf{y}^*}{dt^2} - \frac{d^2\mathbf{y}_0^*}{dt^2} + \left(\mathbf{\Omega}^2 - \frac{d\mathbf{\Omega}}{dt}\right)(\mathbf{y}^* - \mathbf{y}_0^*(t)) - 2\mathbf{\Omega}\left(\frac{d\mathbf{y}^*}{dt} - \frac{d\mathbf{y}_0^*(t)}{dt}\right)$$

Some Transformations under observer changes

Objective (frame indifferent) tensors: map a vector from the observed (inertial) frame back onto the inertial frame

$$\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma}$$

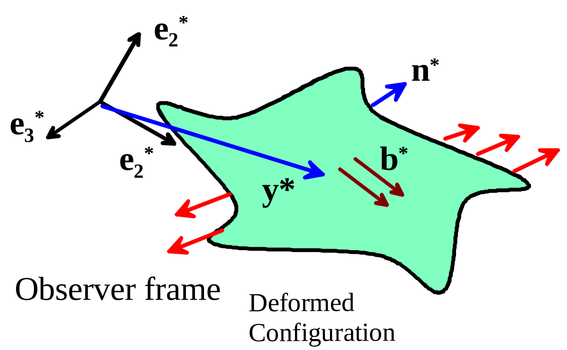
$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T \qquad \mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^T$$



Invariant tensors: map a vector from the reference configuration back onto the reference configuration

$$\mathbf{T}_0 = \mathbf{m} \cdot \boldsymbol{\Sigma}$$

$$\boldsymbol{\Sigma}^* = \boldsymbol{\Sigma}$$



Mixed tensors: map a vector from the reference configuration onto the inertial frame

$$d\mathbf{y} = \mathbf{F}\,d\mathbf{x}$$

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}$$

Some Transformations under observer changes

- **The deformation mapping** transforms as $\mathbf{y}^*(\mathbf{X},t) = \mathbf{y}_0^*(t) + \mathbf{Q}(t)\left(\mathbf{y}(\mathbf{X},t) - \mathbf{y}_0\right)$
- **The deformation gradient** transforms as $\mathbf{F}^* = \dfrac{\partial \mathbf{y}^*}{\partial \mathbf{X}} = \mathbf{Q}\dfrac{\partial \mathbf{y}}{\partial \mathbf{X}} = \mathbf{QF}$
- **The right Cauchy Green strain Lagrange strain, the right stretch tensor are invariant**

$$\mathbf{C}^* = \mathbf{F}^{*T}\mathbf{F}^* = \mathbf{F}^T\mathbf{Q}^T\mathbf{QF} = \mathbf{C} \qquad \mathbf{E}^* = \mathbf{E} \qquad \mathbf{U}^* = \mathbf{U}$$

- **The left Cauchy Green strain, Eulerian strain, left stretch tensor are frame indifferent**

$$\mathbf{B}^* = \mathbf{F}^*\mathbf{F}^{*T} = \mathbf{QFF}^T\mathbf{Q}^T = \mathbf{QCQ}^T \qquad \mathbf{V}^* = \mathbf{QVQ}^T$$

- **The velocity gradient and spin tensor** transform as

$$\mathbf{L}^* = \dot{\mathbf{F}}^*\mathbf{F}^{*-1} = \left(\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}}\right)\mathbf{F}^{-1}\mathbf{Q}^T = \mathbf{QLQ}^T + \mathbf{\Omega}$$

$$\mathbf{W}^* = (\mathbf{L}^* - \mathbf{L}^{*T})/2 = \mathbf{QWQ}^T + \mathbf{\Omega}$$

- **The velocity and acceleration vectors** transform as

$$\mathbf{v}^* = \mathbf{Qv} = \mathbf{Q}\frac{d\mathbf{y}}{dt} = \mathbf{Q}\frac{d}{dt}\mathbf{Q}^T(\mathbf{y}^* - \mathbf{y}_0^*(t)) = \frac{d\mathbf{y}^*}{dt} - \frac{d\mathbf{y}_0^*}{dt} - \mathbf{\Omega}(\mathbf{y}^* - \mathbf{y}_0^*(t))$$

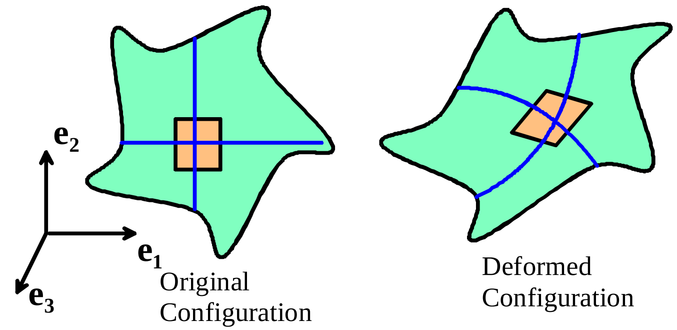
$$\mathbf{a}^* = \mathbf{Qa} = \mathbf{Q}\frac{d^2\mathbf{y}}{dt^2} = \mathbf{Q}\frac{d^2}{dt^2}\mathbf{Q}^T(\mathbf{y}^* - \mathbf{y}_0^*(t)) = \frac{d^2\mathbf{y}^*}{dt^2} - \frac{d^2\mathbf{y}_0^*}{dt^2} + \left(\mathbf{\Omega}^2 - \frac{d\mathbf{\Omega}}{dt}\right)(\mathbf{y}^* - \mathbf{y}_0^*(t)) - 2\mathbf{\Omega}\left(\frac{d\mathbf{y}^*}{dt} - \frac{d\mathbf{y}_0^*(t)}{dt}\right)$$

(the additional terms in the acceleration can be interpreted as the centripetal and coriolis accelerations)

- **The Cauchy stress is frame indifferent** $\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T$ (you can see this from the formal definition, or use the fact that the virtual power must be invariant under a frame change)
- **The material stress is frame invariant** $\mathbf{\Sigma}^* = \mathbf{\Sigma}$
- **The nominal stress transforms as** $\mathbf{S}^* = J(\mathbf{QF})^{-1}\cdot\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T = J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\mathbf{Q}^T = \mathbf{SQ}^T$ (note that this transformation rule will differ if the nominal stress is defined as the transpose of the measure used here…)

Constitutive Laws

Equations relating internal force measures to deformation measures are known as *Constitutive Relations*



General Assumptions:

1. Local homogeneity of deformation
   (a deformation gradient can always be calculated)
2. Principle of local action
   (stress at a point depends on deformation in a vanishingly small material element surrounding the point)

Restrictions on constitutive relations:

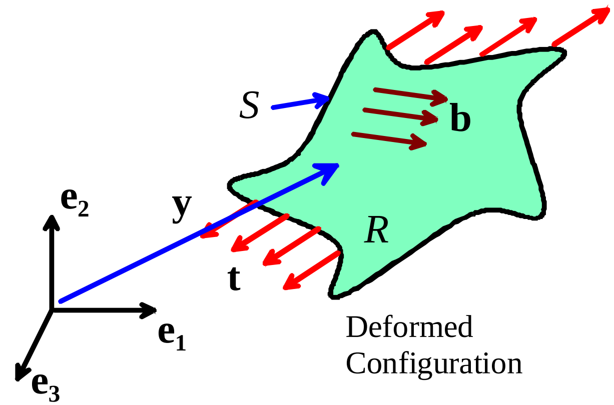
1. Material Frame Indifference – stress-strain relations must transform consistently under a change of observer
2. Constitutive law must always satisfy the second law of thermodynamics for any possible deformation/temperature history.

$$\sigma_{ij} D_{ij} - \frac{1}{\theta} q_i \frac{\partial \theta}{\partial y_i} - \rho \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0$$

Fluids

Properties of fluids

- No natural reference configuration
- Support no shear stress when at rest



Kinematics

- Only need variables that don't depend on ref. config

$$L_{ij} = \frac{\partial v_i}{\partial y_j} \qquad L_{ij} = D_{ij} + W_{ij} \qquad D_{ij} = (L_{ij} + L_{ji})/2 \quad W_{ij} = (L_{ij} - L_{ji})/2 \quad \omega_i = \in_{ijk} \frac{\partial v_k}{\partial y_j} = -\in_{ijk} W_{ij}$$

$$a_i = \left.\frac{\partial v_i}{\partial t}\right|_{x_k=const} = \frac{\partial v_i}{\partial y_k}\frac{\partial y_k}{\partial t} + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} = L_{ik}v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} = \left(D_{ik} + W_{ik}\right)v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const}$$

$$= \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + 2W_{ik}v_k = \left.\frac{\partial v_i}{\partial t}\right|_{y_k=const} + \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + \in_{ijk} \omega_j v_k$$

Conservation Laws

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho D_{kk} = 0 \quad \text{or} \quad \left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

$$\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho\left(\frac{\partial v_i}{\partial y_k}v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const}\right) \qquad \sigma_{ij} = \sigma_{ji}$$

$$\sigma_{ij}D_{ij} - \frac{1}{\theta}q_i\frac{\partial \theta}{\partial y_i} - \rho\left(\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t}\right) \geq 0$$

$$\rho\left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \sigma_{ij}D_{ij} - \frac{\partial q_i}{\partial y_i} + q$$

General Constitutive Models for Fluids

Objectivity and dissipation inequality show that constitutive relations must have form

Internal Energy $\varepsilon = \hat{\varepsilon}(\rho, \theta)$
Entropy $s = \hat{s}(\rho, \theta)$
Free Energy $\psi = \hat{\psi}(\rho, \theta) = \varepsilon - \theta s$
Stress response function $\sigma_{ij} = \hat{\sigma}_{ij}(\theta, \rho, D_{ij}) = -\hat{\pi}_{eq}(\rho, \theta)\delta_{ij} + \hat{\sigma}_{ij}^{vis}(\rho, \theta, D_{ij})$
Heat flux response function $q_i = \hat{q}_i\left(\theta, \rho, \frac{\partial \theta}{\partial y_i}, D_{ij}\right)$

In addition, the constitutive relations must satisfy

$$\hat{\pi}_{eq} = \rho^2 \frac{\partial \hat{\psi}}{\partial \rho} \qquad \hat{s} = -\frac{\partial \psi}{\partial \theta}$$

$$\frac{\partial \hat{\pi}_{eq}}{\partial \theta} = -\rho^2 \frac{\partial \hat{s}}{\partial \rho} \qquad \hat{\pi}_{eq} = \theta \frac{\partial \hat{\pi}_{eq}}{\partial \theta} + \rho^2 \frac{\partial \hat{\varepsilon}}{\partial \rho}$$

$$c_v = -\theta \frac{\partial^2 \hat{\psi}}{\partial \theta^2} \qquad \frac{\partial c_v}{\partial \rho} = -\frac{\theta}{\rho^2} \frac{\partial^2 \hat{\pi}_{eq}}{\partial \theta^2} \qquad \sigma_{ij}^{vis}(\rho, \theta, D_{ij}) D_{ij} \geq 0 \qquad q_i\left(\rho, \theta, \frac{\partial \theta}{\partial y_i}\right) \frac{\partial \theta}{\partial y_i} \geq 0$$

where $c_v(\theta, \rho) = \frac{\partial \hat{\varepsilon}}{\partial \theta}$

Constitutive Models for Fluids

$$\psi = \hat{\psi}(\rho, \theta) = \varepsilon - \theta s \qquad \sigma_{ij} = \hat{\sigma}_{ij}(\theta, \rho, D_{ij}) = -\hat{\pi}_{eq}(\rho, \theta)\delta_{ij} + \hat{\sigma}_{ij}^{vis}(\rho, \theta, D_{ij})$$

Elastic Fluid

$$\psi = \hat{\psi}(\rho) \qquad \sigma_{ij} = -\pi_{eq}(\rho)\delta_{ij}$$

Ideal Gas

$$\varepsilon = c_v\theta = \frac{p}{(\gamma - 1)\rho} \qquad \psi = c_v\theta - \theta\left(c_v \log\theta - R\log\rho - s_0\right) \qquad \sigma_{ij} = -p\delta_{ij} = -\rho R\theta\delta_{ij}$$

Newtonian Viscous

$$\psi = \hat{\psi}(\rho, \theta) \qquad \sigma_{ij} = -(\pi_{eq}(\rho, \theta) - \kappa(\rho, \theta)D_{kk})\delta_{ij} + 2\eta(\rho, \theta)(D_{ij} - D_{kk}\delta_{ij}/3)$$

Non-Newtonian

$$\psi = \hat{\psi}(\rho, \theta)$$

$$\sigma_{ij} = -\pi_{eq}(\rho, \theta)\delta_{ij} + \eta_1(I_1, I_2, I_3, \rho, \theta)\delta_{ij} + \eta_2(I_1, I_2, I_3, \rho, \theta)D_{ij} + \eta_3(I_1, I_2, I_3, \rho, \theta)D_{ik}D_{kj}$$

Derived Field Equations for Newtonian Fluids

Unknowns: p, v_i

Must always satisfy mass conservation $\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho D_{kk} = 0$ or $\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0$

Combine BLM $\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho a_i$ $\quad a_i = \frac{\partial v_i}{\partial y_k} v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} = \left.\frac{\partial v_i}{\partial t}\right|_{y_k=const} + \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + \in_{ijk} \omega_j v_k$

With constitutive law. Also recall $D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right)$

Compressible Navier-Stokes $-\frac{\partial p}{\partial y_i} + 2\frac{\partial}{\partial y_j}\eta(\rho,\theta)\left(D_{ij} - D_{kk}\delta_{ij}/3\right) + \rho b_i = \rho a_i$ $\quad p = \pi_{eq}(\rho,\theta) - \kappa(\rho,\theta) D_{kk}$

With density indep viscosity $-\frac{1}{\rho}\frac{\partial \pi_{eq}}{\partial y_i} + \frac{\eta}{\rho}\frac{\partial^2 v_i}{\partial y_j \partial y_j} + \left(\frac{\kappa}{\rho} - \frac{2\eta}{3\rho}\right)\frac{\partial^2 v_j}{\partial y_j \partial y_i} + b_i = a_i$

For an incompressible Newtonian viscous fluid $-\frac{1}{\rho}\frac{\partial p}{\partial y_i} + \frac{\eta}{\rho}\frac{\partial^2 v_i}{\partial y_j \partial y_j} + b_i = a_i$

Incompressibility reduces mass balance to $\frac{\partial v_i}{\partial y_i} = 0$

For an elastic fluid (Euler eq) $-\frac{\partial \pi_{eq}}{\partial y_i} + \rho b_i = \rho \left.\frac{\partial v_i}{\partial t}\right|_{y_k=const} + \frac{1}{2}\rho\frac{\partial}{\partial y_i}(v_k v_k) + \rho \in_{ijk} \omega_j v_k$

Derived Field Equations for Fluids

Recall vorticity vector $\omega_i = \in_{ijk} \frac{\partial v_k}{\partial y_j} = -\in_{ijk} W_{ij}$ $\qquad \in_{ijk} \frac{\partial a_k}{\partial y_j} = \left.\frac{\partial \omega_i}{\partial t}\right|_{\mathbf{x}=const} - D_{ij}\omega_j + \frac{\partial v_k}{\partial y_k}\omega_i$

Vorticity transport equation (constant temperature, density independent viscosity)

$$+\frac{\eta}{\rho}\frac{\partial^2 \omega_i}{\partial y_j \partial y_j} - \frac{1}{\rho^2}\in_{ijk}\frac{\partial \rho}{\partial y_j}\left\{\eta \frac{\partial^2 v_k}{\partial y_l \partial y_l} + \left(\kappa - \frac{2\eta}{3}\right)\frac{\partial^2 v_l}{\partial y_l \partial y_k}\right\} + \in_{ijk}\frac{\partial}{\partial y_j}(b_k) + D_{ij}\omega_j - \frac{\partial v_k}{\partial y_k}\omega_i = \left.\frac{\partial \omega_i}{\partial t}\right|_{\mathbf{x}=const}$$

For an elastic fluid $\qquad \in_{ijk}\frac{\partial}{\partial x_j}(b_k) + D_{ij}\omega_j - \frac{\partial v_k}{\partial y_k}\omega_i = \left.\frac{\partial \omega_i}{\partial t}\right|_{\mathbf{x}=const}$

For an incompressible fluid $\qquad +\frac{\eta}{\rho}\frac{\partial^2 \omega_i}{\partial y_j \partial y_j} + \in_{ijk}\frac{\partial}{\partial x_j}(b_k) + D_{ij}\omega_j = \left.\frac{\partial \omega_i}{\partial t}\right|_{\mathbf{x}=const}$

If flow of an ideal fluid is irrotational at *t=0* and body forces are curl free, then flow remains irrotational for all time (**Potential flow)**

Derived field equations for fluids

For an elastic fluid

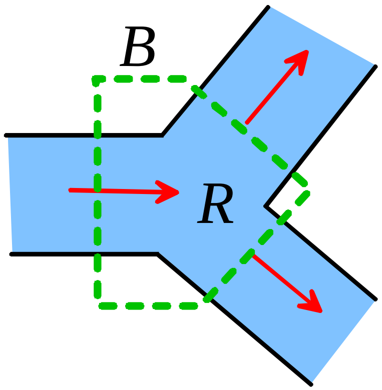
- Bernoulli $$H = \psi + \frac{\pi_{eq}}{\rho} + \frac{1}{2} v_i v_i + \Phi = \text{constant}$$ along streamline

For irrotational flow $$H = \psi + \frac{\pi_{eq}}{\rho} + \frac{1}{2} v_i v_i + \Phi = \text{constant}$$ everywhere

For incompressible fluid $$\frac{p}{\rho} + \frac{1}{2} v_i v_i + \Phi = \text{constant}$$

Solving fluids problems: control volume approach

Governing equations for a control volume (review)



- **Mass Conservation:** $\frac{d}{dt}\int_R \rho dV + \int_B \rho \mathbf{v}\cdot\mathbf{n} dA = 0$
- **Linear Momentum Balance** $\int_B \mathbf{n}\cdot\boldsymbol{\sigma} dA + \int_R \rho\mathbf{b} dV = \frac{d}{dt}\int_R \rho\mathbf{v} dV + \int_B (\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n} dA$
- **Angular Momentum Balance** $\int_B \mathbf{y}\times(\mathbf{n}\cdot\boldsymbol{\sigma}) dA + \int_R \mathbf{y}\times(\rho\mathbf{b}) dA = \frac{d}{dt}\int_R \mathbf{y}\times\rho\mathbf{v} dV + \int_B (\mathbf{y}\times\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n} dA$
- **Mechanical Power Balance**

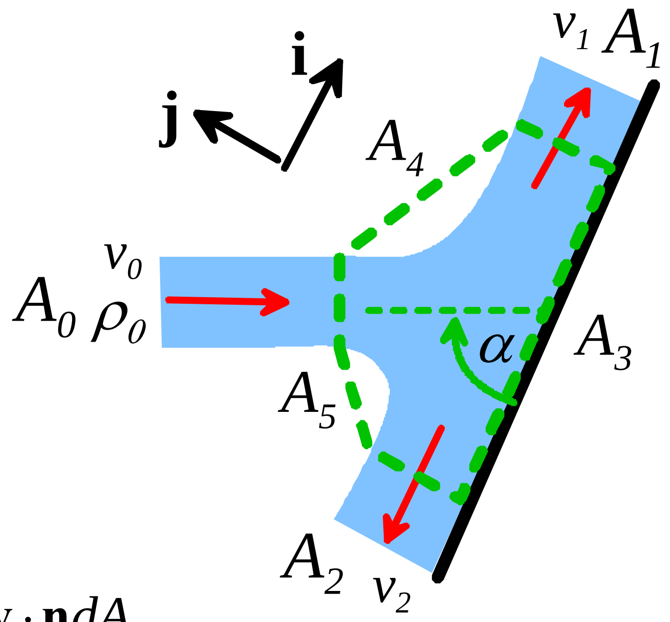
$$\int_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v} dA + \int_R \rho\mathbf{b}\cdot\mathbf{v} dV = \int_R \boldsymbol{\sigma} : \mathbf{D} dV + \frac{d}{dt}\int_R \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v}) dV + \int_B \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})\mathbf{v}\cdot\mathbf{n} dA$$

- **First law of thermodynamics**

$$\int_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v} dA + \int_R \rho\mathbf{b}\cdot\mathbf{v} dV - \int_B \mathbf{q}\cdot\mathbf{n} dA + \int_V q dV = \frac{d}{dt}\int_R \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) dV + \int_B \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\cdot\mathbf{n} dA$$

Example

Steady 2D flow, ideal fluid
Calculate the force acting on the wall
Take surrounding pressure to be zero



$$\int_B \mathbf{n}\cdot\boldsymbol{\sigma}\,dA + \int_R \rho\mathbf{b}\,dV = \frac{d}{dt}\int_R \rho\mathbf{v}\,dV + \int_B (\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}\,dA$$

$$\int_A (-p_0\mathbf{n}\cdot\mathbf{j}\,dA)\mathbf{j} - A_0\rho_0 v_0^2 \sin\alpha\,\mathbf{j} + \int_{A_3} (p - p_0)\,dA\,\mathbf{j} = 0$$

$$\mathbf{F} = -A_0\rho_0 v_0^2 \sin\alpha\,\mathbf{j}$$

Exact solutions: potential flow

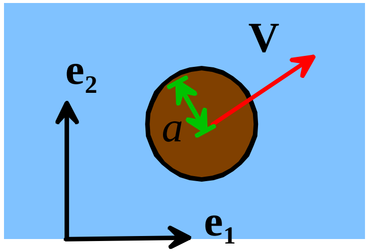
If flow irrotational at *t=0*, remains irrotational; Bernoulli holds everywhere

Irrotational: curl(**v)=0** so $$v_i = \frac{\partial \Omega}{\partial y_i}$$

Mass cons $$\frac{\partial v_i}{\partial y_i} = 0 \Rightarrow \frac{\partial^2 \Omega}{\partial y_i \partial y_i} = 0$$

Bernoulli $$\frac{p}{\rho} + \frac{1}{2} v_i v_i + \Phi + \frac{\partial \Omega}{\partial t} = \text{constant}$$

Example: flow surrounding a moving sphere

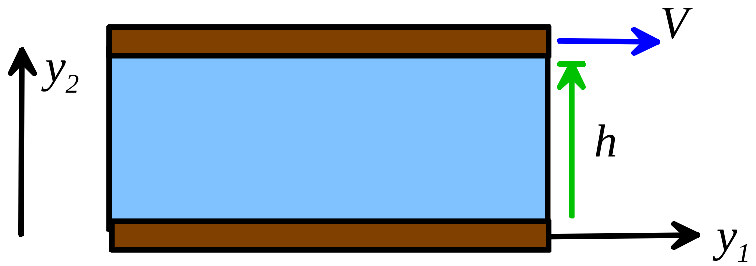


$$\Omega = -\frac{a^2 V_\alpha (y_\alpha - V_\alpha t)}{r^2} \qquad r = \sqrt{(y_\alpha - V_\alpha t)(y_\alpha - V_\alpha t)}$$

Exact solutions: Stokes Flow

Steady laminar viscous flow between plates
Assume constant pressure gradient in horizontal direction

$$-\frac{1}{\rho}\frac{\partial p}{\partial y_i}+\frac{\eta}{\rho}\frac{\partial^2 v_i}{\partial y_j \partial v_j}+b_i \approx \left.\frac{\partial v_i}{\partial t}\right|_{y_k=const} \Rightarrow -\frac{\Delta p}{\Delta L}+\eta\frac{\partial^2 f}{\partial y_2^2}=0$$



Solve subject to boundary conditions

$$\mathbf{v}=\left[V\frac{y_2}{h}-\frac{\Delta p}{2\Delta L}y_2(h-y_2)\right]\mathbf{e}_1$$

$$\boldsymbol{\sigma}=\frac{\eta V}{h}+\frac{\Delta p}{\Delta L}\left(\frac{h}{2}-y_2\right)$$

Exact Solutions: Acoustics

Assumptions:

- Small amplitude pressure and density fluctuations
- Irrotational flow
- Negligible heat flow
- Neglect body forces

Irrotational: $v_i = \dfrac{\partial \Omega}{\partial y_i}$

Approximate N-S as: $-\dfrac{\partial p}{\partial y_i} \approx \rho \left.\dfrac{\partial v_i}{\partial t}\right|_{y_k=const}$ $\quad -\dfrac{\partial^2 p}{\partial y_i \partial y_i} \approx \rho \left.\dfrac{\partial^2 v_i}{\partial y_i \partial t}\right|_{y_k=const}$ $\quad -\dfrac{\partial^2 p}{\partial y_i \partial t} \approx \rho \left.\dfrac{\partial^2 v_i}{\partial t^2}\right|_{y_k=const}$

For small perturbations: $\dfrac{\partial p}{\partial t} = c_s^2 \dfrac{\partial \delta\rho}{\partial t}$ $\quad c_s = \sqrt{\left.\dfrac{\partial p}{\partial \rho}\right|_{s=const}}$

Mass conservation: $\left.\dfrac{\partial \delta\rho}{\partial t}\right|_{\mathbf{x}=const} + \rho \dfrac{\partial v_i}{\partial y_i} = 0$

Combine: $\boxed{\dfrac{\partial^2 \Omega}{\partial t^2} - c_s^2 \dfrac{\partial^2 \Omega}{\partial y_i \partial y_j} = 0 \qquad \delta p = -\rho_0 \dfrac{\partial \Omega}{\partial t}}$

(Wave equation)

Wave speed in an ideal gas

Assume heat flow can be neglected

Entropy equation: $$\theta \left. \frac{\partial s}{\partial t} \right|_{\mathbf{x}=const} = -\frac{\partial q_i}{\partial y_i} + q \Rightarrow s = const$$

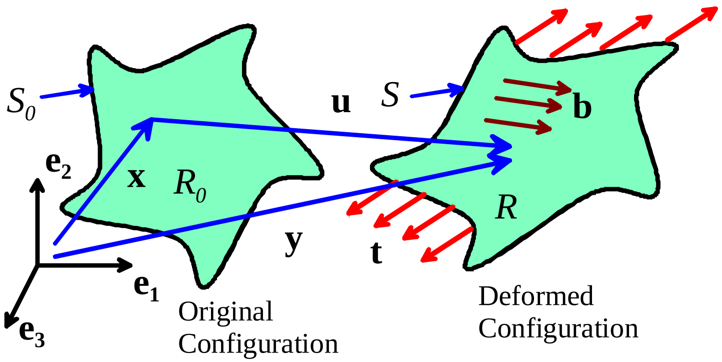
$$\varepsilon = c_v \theta \qquad \psi = c_v\theta - \theta\left(c_v \log\theta - R\log\rho + s_0\right) \qquad p = \rho R \theta$$

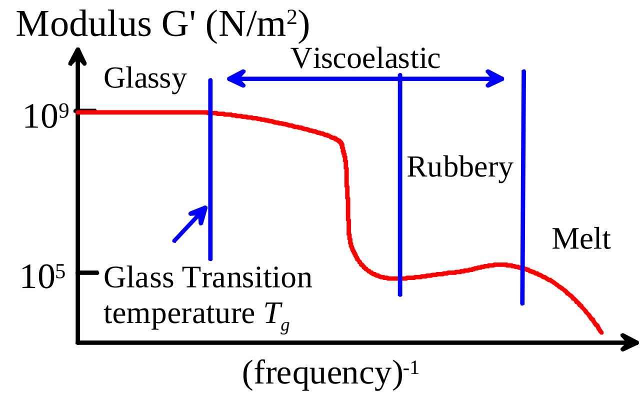
$$s = \left(c_v \log\theta - R\log\rho + s_0\right) \Rightarrow \theta = \rho^{R/c_v} \exp[(s - s_0)/c_v)$$

$$R / c_v = \gamma - 1 \qquad \text{so} \qquad p = k\rho^{\gamma}$$

Hence: $$c_s = \sqrt{\left. \frac{\partial p}{\partial \rho} \right|_{s=const}} = \sqrt{k\gamma\rho^{\gamma-1}} = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma R \theta}$$

Application of continuum mechanics to elasticity



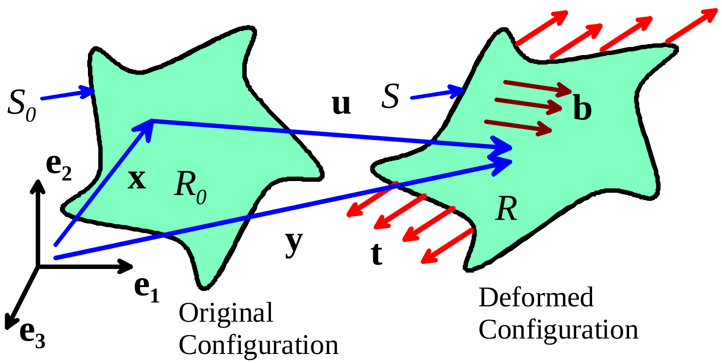


Material characterized by

- The mass density ρ_0 per unit reference volume
- The specific internal energy ε
- The specific entropy s
- The specific Helmholtz free energy $\psi = \varepsilon - \theta s$

- A stress response function, e.g. $\Sigma_{ij} = \hat{\Sigma}_{ij}(\theta,$ kinematic and internal variables$)$. Here, Σ_{ij} is the material stress – one can use response functions for other stress measures as well.
- A heat flux response function $q_i = \hat{q}_i(\theta,$ kinematic and internal variables$)$. In actual calculations for solids it is often preferable to define a heat fluxe response function that characterizes heat flow through the reference configuration – an appropriate measure is defined below.

General structure of constitutive relations



$$F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$$

$$C_{ij} = F_{ki} F_{kj}$$

$$B_{ij} = F_{ik} F_{jk}$$

$$Q_i = J F_{ik}^{-1} q_k$$

$$\mathbf{S} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \qquad S_{ij} = J F_{ik}^{-1} \sigma_{kj}$$

$$\boldsymbol{\Sigma} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T} \quad \Sigma_{ij} = J F_{ik}^{-1} \sigma_{kl} F_{jl}^{-1}$$

$$\frac{\partial S_{ji}}{\partial x_j} + \rho_0 b_i = \rho_0 \left.\frac{\partial v_i}{\partial t}\right|_{x_i = const}$$

$$\frac{1}{2}\Sigma_{ij}\dot{C}_{ij} - \frac{1}{\theta} Q_i \frac{\partial \theta}{\partial x_i} - \rho_0 \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0$$

To be consistent with frame indifference and the laws of thermodynamics, the specific free energy, internal energy, Helmholtz free energy, stress response function and heat transfer function must have the forms

- Specific internal energy $\varepsilon = \hat{\varepsilon}(\mathbf{C}, \theta)$
- Specific entropy $s = \hat{s}(\mathbf{C}, \theta)$
- Specific Helmholtz free energy $\psi = \hat{\psi}(\mathbf{C}, \theta) = \varepsilon - \theta s$
- Stress response function $\Sigma_{ij} = \hat{\Sigma}_{ij}(\mathbf{C}, \theta)$
- Heat flux response function $Q_i = \hat{Q}_i\left(\theta, \mathbf{C}, \frac{\partial \theta}{\partial y_i}\right)$

$$\mathbf{F}^* = \mathbf{QF}$$

$$\mathbf{B}^* = \mathbf{F}^* \mathbf{F}^{*T} = \mathbf{QBQ}^T$$

$$\mathbf{C}^* = \mathbf{F}^{*T} \mathbf{F}^* = \mathbf{C}$$

$$\boldsymbol{\Sigma}^* = \boldsymbol{\Sigma}$$

Frame indifference, dissipation inequality

$$\hat{\Sigma}_{ij} = 2\rho_0 \frac{\partial \hat{\psi}}{\partial C_{ij}} \qquad \hat{s} = -\frac{\partial \hat{\psi}}{\partial \theta}$$

$$\frac{\partial \hat{\Sigma}_{ij}}{\partial \theta} = -2\rho_0 \frac{\partial \hat{s}}{\partial C_{ij}}$$

$$Q_k \frac{\partial \theta}{\partial x_k} \leq 0$$

Forms of constitutive relation used in literature

$$I_1 = \mathrm{trace}(\mathbf{B}) = B_{kk}$$

$$I_2 = \frac{1}{2}\left(I_1^2 - \mathbf{B}\cdot\cdot\mathbf{B}\right) = \frac{1}{2}\left(I_1^2 - B_{ik}B_{ki}\right)$$

$$I_3 = \det\mathbf{B} = J^2$$

$$\overline{I}_1 = \frac{I_1}{J^{2/3}} = \frac{B_{kk}}{J^{2/3}}$$

$$\overline{I}_2 = \frac{I_2}{J^{4/3}} = \frac{1}{2}\left(\overline{I}_1^2 - \frac{\mathbf{B}\cdot\cdot\mathbf{B}}{J^{4/3}}\right) = \frac{1}{2}\left(\overline{I}_1^2 - \frac{B_{ik}B_{ki}}{J^{4/3}}\right)$$

$$J = \sqrt{\det\mathbf{B}}$$

$$\mathbf{B} = \lambda_1^2 \mathbf{b}^{(1)} \otimes \mathbf{b}^{(1)} + \lambda_2^2 \mathbf{b}^{(2)} \otimes \mathbf{b}^{(2)} + \lambda_3^2 \mathbf{b}^{(3)} \otimes \mathbf{b}^{(3)}$$

- Strain energy potential $W = \rho_0 \psi$

$$W(\mathbf{F}) = \hat{W}(\mathbf{C}) = U(I_1, I_2, I_3) = \overline{U}(\overline{I}_1, \overline{I}_2, J) = \tilde{U}(\lambda_1, \lambda_2, \lambda_3)$$

$$\sigma_{ij} = \frac{1}{J} F_{ik} \frac{\partial W}{\partial F_{jk}}$$

$$\sigma_{ij} = \frac{2}{\sqrt{I_3}}\left[\left(\frac{\partial U}{\partial I_1} + I_1 \frac{\partial U}{\partial I_2}\right) B_{ij} - \frac{\partial U}{\partial I_2} B_{ik} B_{kj}\right] + 2\sqrt{I_3}\frac{\partial U}{\partial I_3}\delta_{ij}$$

$$\sigma_{ij} = \frac{2}{J}\left[\frac{1}{J^{2/3}}\left(\frac{\partial \overline{U}}{\partial \overline{I}_1} + \overline{I}_1 \frac{\partial \overline{U}}{\partial \overline{I}_2}\right) B_{ij} - \left(\overline{I}_1 \frac{\partial \overline{U}}{\partial \overline{I}_1} + 2\overline{I}_2 \frac{\partial \overline{U}}{\partial \overline{I}_2}\right)\frac{\delta_{ij}}{3} - \frac{1}{J^{4/3}} \frac{\partial \overline{U}}{\partial \overline{I}_2} B_{ik} B_{kj}\right] + \frac{\partial \overline{U}}{\partial J}\delta_{ij}$$

$$\sigma_{ij} = \frac{\lambda_1}{\lambda_1\lambda_2\lambda_3}\frac{\partial \tilde{U}}{\partial \lambda_1} b_i^{(1)} b_j^{(1)} + \frac{\lambda_2}{\lambda_1\lambda_2\lambda_3}\frac{\partial \tilde{U}}{\partial \lambda_2} b_i^{(2)} b_j^{(2)} + \frac{\lambda_3}{\lambda_1\lambda_2\lambda_3}\frac{\partial \tilde{U}}{\partial \lambda_3} b_i^{(3)} b_j^{(3)}$$

Specific forms for free energy function

- Neo-Hookean material

$$\bar{U} = \frac{\mu_1}{2}(\bar{I}_1 - 3) + \frac{K_1}{2}(J-1)^2$$

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}}\left(B_{ij} - \frac{1}{3}B_{kk}\delta_{ij}\right) + K_1(J-1)\delta_{ij}$$

- Mooney-Rivlin

$$\bar{U} = \frac{\mu_1}{2}(\bar{I}_1 - 3) + \frac{\mu_2}{2}(\bar{I}_2 - 3) + \frac{K_1}{2}(J-1)^2$$

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}}\left(B_{ij} - \frac{1}{3}B_{kk}\delta_{ij}\right) + \frac{\mu_2}{J^{7/3}}\left(B_{kk}B_{ij} - \frac{1}{3}[B_{kk}]^2\delta_{ij} - B_{ik}B_{kj} + \frac{1}{3}B_{kn}B_{nk}\delta_{ij}\right) + K_1(J-1)\delta_{ij}$$

- Generalized polynomial function

$$\bar{U} = \sum_{i+j=1}^{N} C_{ij}(\bar{I}_1 - 3)^i(\bar{I}_2 - 3)^j + \sum_{i=1}^{N}\frac{K_i}{2}(J-1)^{2i}$$

- Ogden

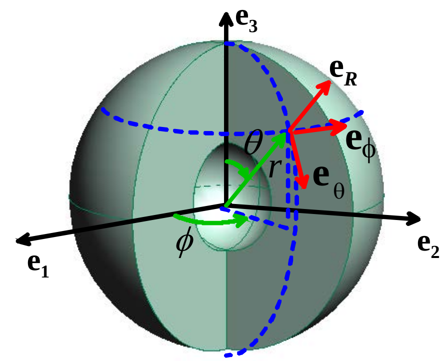
$$\tilde{U} = \sum_{i=1}^{N}\frac{2\mu_i}{\alpha_i^2}(\bar{\lambda}_1^{\alpha_i} + \bar{\lambda}_2^{\alpha_i} + \bar{\lambda}_3^{\alpha_i} - 3) + \frac{K_1}{2}(J-1)^2$$

- Arruda-Boyce

$$\bar{U} = \mu\left\{\frac{1}{2}(\bar{I}_1 - 3) + \frac{1}{20\beta^2}(\bar{I}_1^2 - 9) + \frac{11}{1050\beta^4}(\bar{I}_1^3 - 27) + \dots\right\} + \frac{K}{2}(J-1)^2$$

Solving problems for elastic materials (spherical/axial symmetry)

- Assume incompressiblility



- Kinematics

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{bmatrix} \qquad \mathbf{F} \equiv \begin{bmatrix} F_{rr} & 0 & 0 \\ 0 & F_{\theta\theta} & 0 \\ 0 & 0 & F_{\phi\phi} \end{bmatrix} \qquad \mathbf{B} \equiv \begin{bmatrix} B_{rr} & 0 & 0 \\ 0 & B_{\theta\theta} & 0 \\ 0 & 0 & B_{\phi\phi} \end{bmatrix}$$

$$F_{rr} = \frac{dr}{dR} \quad F_{\phi\phi} = F_{\theta\theta} = \frac{r}{R} \qquad B_{rr} = \left(\frac{dr}{dR}\right)^2 \quad B_{\phi\phi} = B_{\theta\theta} = \left(\frac{r}{R}\right)^2$$

$$\left(\frac{dr}{dR}\right)\left(\frac{r}{R}\right)^2 = 1 \qquad r^3 - a^3 = R^3 - A^3$$

- Constitutive law

$$\sigma_{rr} = 2\left[\left(\frac{\partial U}{\partial I_1} + I_1 \frac{\partial U}{\partial I_2}\right)B_{rr} - \frac{I_1}{3}\frac{\partial U}{\partial I_1} - \frac{2I_2}{3}\frac{\partial U}{\partial \bar{I}_2} - \frac{\partial U}{\partial I_2}B_{rr}^2\right] + p$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = 2\left[\left(\frac{\partial U}{\partial I_1} + I_1 \frac{\partial U}{\partial I_2}\right)B_{\theta\theta} - \frac{I_1}{3}\frac{\partial U}{\partial I_1} - \frac{2I_2}{3}\frac{\partial U}{\partial \bar{I}_2} - \frac{\partial U}{\partial I_2}B_{\theta\theta}^2\right] + p$$

- Equilibrium (or use PVW) $\dfrac{d\sigma_{rr}}{dr} + \dfrac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + \rho_0 b_r = 0$ (gives ODE for p(r))

- Boundary conditions

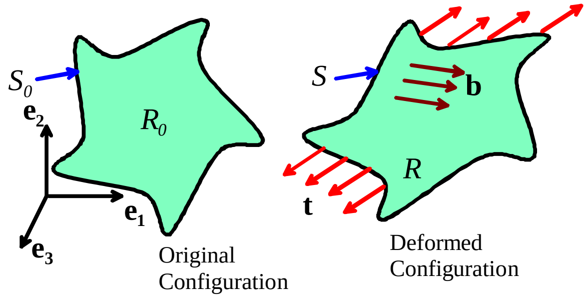
$$u_r(a) = g_a \qquad u_r(b) = g_b$$

$$\sigma_{rr}(a) = t_a \qquad \sigma_{rr}(b) = t_b$$

Linearized field equations for elastic materials

Approximations:

- **Linearized kinematics**
- **All stress measures equal**
- **Linearize stress-strain relation**



$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad \sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \alpha_{kl}\Delta\theta) \qquad \frac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j = \rho \frac{\partial^2 u_j}{\partial t^2}$$

$$u_i = u_i^*(t) \quad \text{on} \quad \partial_1 R \qquad \sigma_{ij} n_i = t_j^*(t) \quad \text{on} \quad \partial_2 R$$

Elastic constants related to strain energy/unit vol

$$C_{ijkl} = \frac{\partial \hat{\sigma}_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 U}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \qquad \beta_{ij} = \frac{\partial \hat{\sigma}_{ij}}{\partial \theta} = \frac{\partial^2 U}{\partial \varepsilon_{ij} \partial \theta}$$

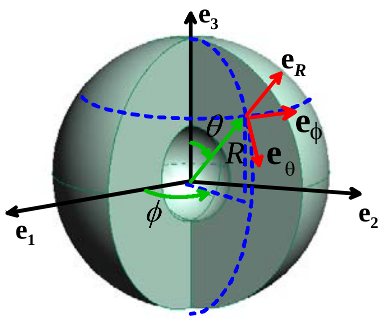
$$\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma} + \boldsymbol{\alpha}\Delta T \qquad \boldsymbol{\sigma} = \mathbf{C}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}\Delta T)$$

Isotropic materials:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha \Delta T \delta_{ij} \qquad \sigma_{ij} = \frac{E}{1+\nu}\left\{ \varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} \right\} - \frac{E\alpha\Delta T}{1-2\nu}\delta_{ij}$$

Solving linear elasticity problems spherical/axial symmetry

$$\sigma \equiv \begin{bmatrix} \sigma_{RR} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{bmatrix} \qquad \varepsilon \equiv \begin{bmatrix} \varepsilon_{RR} & 0 & 0 \\ 0 & \varepsilon_{\theta\theta} & 0 \\ 0 & 0 & \varepsilon_{\phi\phi} \end{bmatrix}$$



- Kinematics $\quad \varepsilon_{RR} = \dfrac{du}{dR} \qquad \varepsilon_{\phi\phi} = \varepsilon_{\theta\theta} = \dfrac{u}{R}$
- Constitutive law

$$\begin{bmatrix} \sigma_{RR} \\ \sigma_{\theta\theta} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 2\nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \dfrac{du}{dR} \\ \dfrac{u}{R} \end{bmatrix} - \frac{E\alpha\Delta T}{1-2\nu} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Equilibrium

$$\frac{d\sigma_{RR}}{dR} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + \rho_0 b_R = 0$$

$$\frac{d^2u}{dR^2} + \frac{2}{R}\frac{du}{dR} - \frac{2u}{R^2} = \frac{d}{dR}\left\{\frac{1}{R^2}\frac{d}{dR}\left(R^2 u\right)\right\} = \frac{\alpha(1+\nu)}{(1-\nu)}\frac{d\Delta T}{dR} - \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\rho_0 b(R)$$

- Boundary conditions

$$u_r(a) = g_a \qquad u_r(b) = g_b$$

$$\sigma_{rr}(a) = t_a \qquad \sigma_{rr}(b) = t_b$$

Some simple static linear elasticity solutions

Navier equation: $$\frac{1}{1-2\nu}\frac{\partial^2 u_k}{\partial x_k \partial x_i}+\frac{\partial^2 u_i}{\partial x_k \partial x_k}+\rho_0\frac{b_i}{\mu}=\frac{\rho_0}{\mu}\frac{\partial^2 u_i}{\partial t^2}$$

Potential Representation (statics): $$u_i=\frac{2(1+\nu)}{E}\left(\Psi_i+\frac{1}{4(1-\nu)}\frac{\partial}{\partial x_i}\left(\phi-x_k\Psi_k\right)\right) \qquad \frac{\partial^2\Psi_i}{\partial x_j\partial x_j}=-\rho_0 b_i \qquad \frac{\partial^2\phi}{\partial x_k\partial x_k}=-\rho_0 b_i x_i$$

Point force in an infinite solid: $$\Psi_i=\frac{P_i}{4\pi R} \qquad \phi=0$$

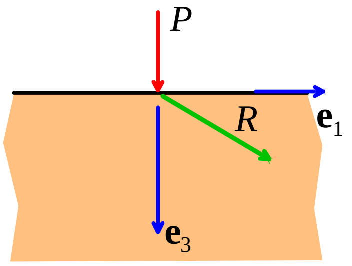
$$u_i=\frac{(1+\nu)}{8\pi E(1-\nu)R}\left\{\frac{P_k x_k x_i}{R^2}+(3-4\nu)P_i\right\}$$

$$\varepsilon_{ij}=\frac{-(1+\nu)}{8\pi E(1-\nu)R^2}\left\{\frac{3P_k x_k x_i x_j}{R^3}-\frac{P_k x_k\delta_{ij}}{R}+(1-2\nu)\frac{P_i x_j+P_j x_i}{R}\right\}$$

$$\sigma_{ij}=\frac{-1}{8\pi(1-\nu)R^2}\left\{\frac{3P_k x_k x_i x_j}{R^3}+(1-2\nu)\frac{P_i x_j+P_j x_i-\delta_{ij}P_k x_k}{R}\right\}$$

Point force normal to a surface:

$$\Psi_i=\frac{(1-\nu)\delta_{i3}}{\pi R} \qquad \phi=-\frac{(1-2\nu)(1-\nu)}{\pi}\log(R+x_3)$$



$$u_i=\frac{(1+\nu)P}{2\pi E}\left\{\frac{x_3 x_i}{R^3}+(3-4\nu)\frac{\delta_{i3}}{R}-\frac{(1-2\nu)}{R+x_3}\left(\delta_{3i}+\frac{x_i}{R}\right)\right\}$$

$$\sigma_{ij}=\frac{P}{2\pi R^2}\left\{-3\frac{x_i x_j x_3}{R^3}+\frac{(1-2\nu)(2R+x_3)}{R(R+x_3)^2}\left(x_i x_j+\delta_{ij}x_3^2-x_3\left(\delta_{i3}x_j+\delta_{j3}x_i\right)\right)+\frac{(1-2\nu)R^2}{(R+x_3)^2}\left(\delta_{i3}\delta_{j3}-\delta_{ij}\right)\right\}$$

Dynamic elasticity solutions

Plane wave solution $$u_i = a_i f(ct - x_k p_k)$$

Navier equation $$\frac{1}{1-2\nu} \frac{\partial^2 u_k}{\partial x_k \partial x_i} + \frac{\partial^2 u_i}{\partial x_k \partial x_k} + \rho_0 \frac{b_i}{\mu} = \frac{\rho_0}{\mu} \frac{\partial^2 u_i}{\partial t^2}$$

$$\left(\mu - \rho_0 c^2\right) a_k + \frac{\mu}{1-2\nu} p_i a_i p_k = 0$$

Solutions: $$a_i p_i = 0 \quad \Rightarrow \quad c^2 = c_2^2 = \rho_0 / \mu$$

$$a_i = \eta p_i \quad \Rightarrow \quad c^2 = c_L^2 = 2\mu(1-\nu) / \rho_0 (1-2\nu)$$