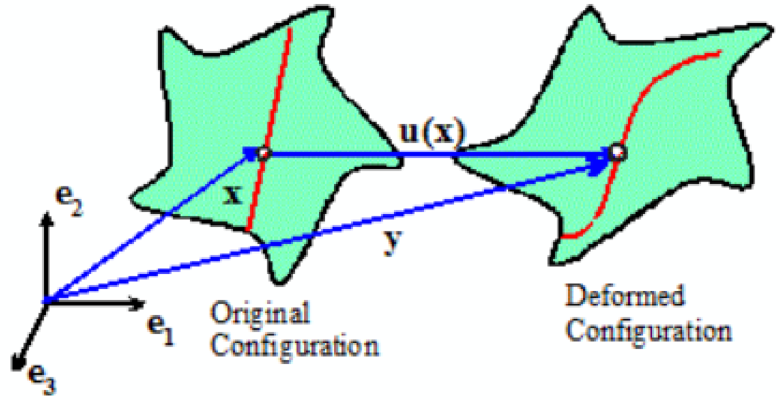
# Infinitesimal Deformations

$\underline{u}$ : $\nabla \underline{u}$ ; $\nabla \underline{u} \ll 1$

This requires both small strains and small rotations




## Infinitesimal Strain Tensor

$$\varepsilon = \text{sym}(\nabla \underline{u})$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Properties:

① $\varepsilon \approx E$

To see this note $E = \frac{1}{2}(F^T F - I)$

$$= \frac{1}{2}\left[(I + \nabla \underline{u})^T (I + \nabla \underline{u}) - I\right]$$

$$= \varepsilon + \underbrace{\frac{1}{2} \nabla \underline{u}^T \nabla \underline{u}}_{\text{neglect}}$$

② $\text{trace}(\varepsilon) = \dfrac{dV - dV_0}{dV_0}$

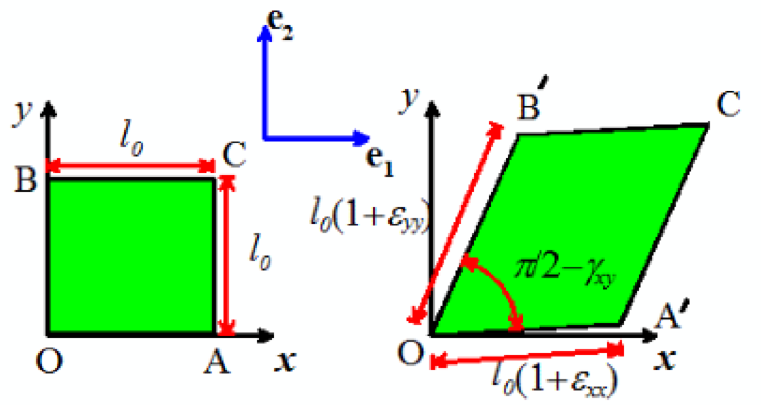
To see this recall $\dfrac{dV}{dV_0} = \det(I + \nabla \underline{u})$

$$= 1 + tr(\nabla \underline{u}) + O(|\nabla \underline{u}|^2)$$

$$= 1 + tr(\varepsilon) + \text{''}$$

$$\frac{dV}{dV_0} - 1 = tr(\varepsilon)$$

(3) Components of $\varepsilon$ related to length & angle changes

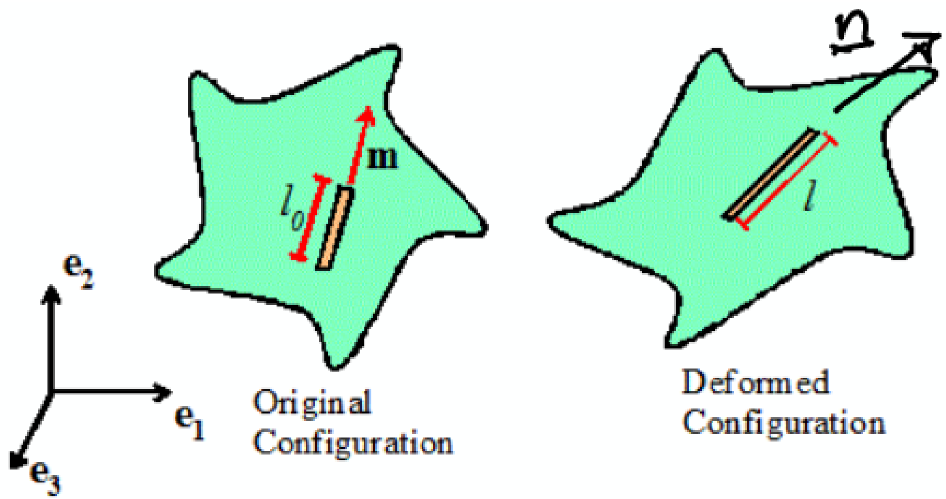



$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix}$$

$\gamma_{xy} = 2\varepsilon_{xy} \leftarrow$ "Engineering" shear strain

④ $\dfrac{l - l_0}{l_0} \approx \underset{\sim}{m} \cdot \varepsilon \underset{\sim}{m}$

$\approx \underset{\sim}{n} \cdot \varepsilon \underset{\sim}{n}$




(5) $\varepsilon$ is symmetric

$\Rightarrow$ real eigenvalues

– principal strains

Orthogonal eigenvectors : principal strain directions

Infinitesimal rotation tensor $\quad W = \text{skew}(\nabla \underset{\sim}{u})$

$$W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

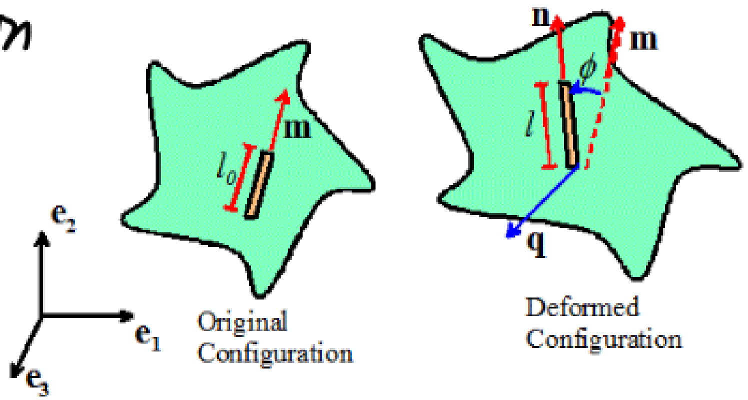
Properties :

① $d\underset{\sim}{x} \, (I + W) \, d\underset{\sim}{x} = d\underset{\sim}{x} \cdot d\underset{\sim}{x}$

Hence $\underset{\sim}{w}$ represents an infinitesimal rotation

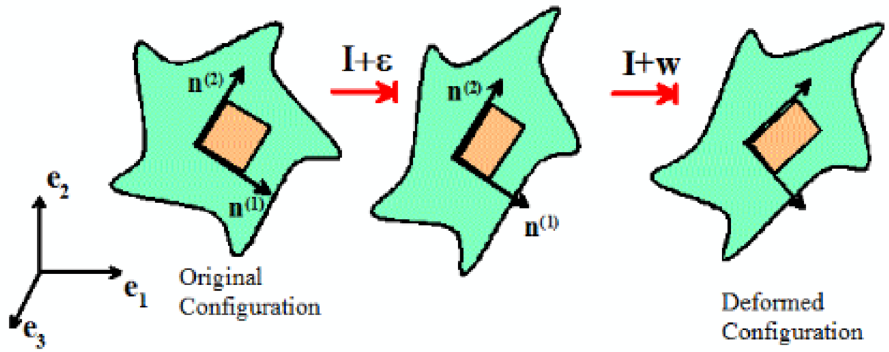
$$\underset{\sim}{w}\, d\underset{\sim}{x} = \sin\theta\, \underset{\sim}{n} \times d\underset{\sim}{x} \qquad \sin\theta\, \underset{\sim}{n} = \mathrm{dual}(\underset{\sim}{w})$$

$\theta$ represents small angle of rotation of average rotation of all fibers passing through a point




## Decomposition of $\nabla \underset{\sim}{u}$

$$\nabla \underset{\sim}{u} = \varepsilon + w$$



# Inverting the strain-displacement relations

Given $\varepsilon_{ij}(\underline{x})$, can we find $u_i(\underline{x})$?

① Rigid motions have $\varepsilon = 0 \Rightarrow$ can only find $\underline{u}$ to within a rigid motion

② 6 $\varepsilon_{ij}$, but only 3 $u_i \Rightarrow \varepsilon_{ij}$ are not all independent

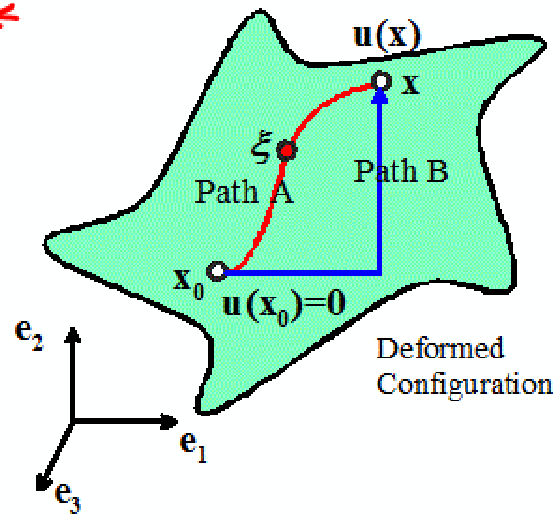
## Compatibility Conditions

$$\varepsilon = \text{sym}(\nabla \underline{u}) \iff \nabla \times \nabla \times \varepsilon = 0$$

or $$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_l} = 0$$

Also $u_i = \int_{\underline{x}_0}^{\underline{x}} \Lambda_{ij}(\underline{x}, \underline{\xi})\, d\xi_j$ *

$$\Lambda_{ij} = \varepsilon_{ij} + (x_k - \xi_k)\left(\frac{\partial \varepsilon_{ij}}{\partial \xi_k} - \frac{\partial \varepsilon_{kj}}{\partial \xi_i}\right)$$




We need to show integral is path independent

For this to be true $\Lambda_{ij}\, d\xi_j$ must be a perfect differential

$$\Rightarrow \Lambda_{ij} = \frac{\partial P_i}{\partial \xi_j} \quad \text{for some } P_i$$

$$\Rightarrow \frac{\partial \Lambda_{ij}}{\partial \xi_n} = \frac{\partial^2 P_i}{\partial \xi_j \partial \xi_n} = \frac{\partial \Lambda_{in}}{\partial \xi_j} \Rightarrow \frac{\partial \Lambda_{ij}}{\partial \xi_n} - \frac{\partial \Lambda_{in}}{\partial \xi_j} = 0$$

$$\Lambda_{ij} = \varepsilon_{ij} + (x_k - \xi_k)\left(\frac{\partial \varepsilon_{ij}}{\partial \xi_k} - \frac{\partial \varepsilon_{kj}}{\partial \xi_i}\right)$$

$$\frac{\partial \Lambda_{ij}}{\partial \xi_n} = \frac{\partial \varepsilon_{ij}}{\partial \xi_n} - \left(\frac{\partial \varepsilon_{ij}}{\partial \xi_n} - \frac{\partial \varepsilon_{nj}}{\partial \xi_i}\right) + (x_k - \xi_k)\left(\frac{\partial \varepsilon_{ij}}{\partial \xi_n \partial \xi_k} - \frac{\partial \varepsilon_{kj}}{\partial \xi_n \partial \xi_i}\right)$$

$$\frac{\partial \Lambda_{in}}{\partial \xi_j} = \frac{\partial \varepsilon_{in}}{\partial \xi_j} - \left(\frac{\partial \varepsilon_{in}}{\partial \xi_j} - \frac{\partial \varepsilon_{nj}}{\partial \xi_i}\right) + (x_k - \xi_k)\left(\frac{\partial \varepsilon_{in}}{\partial \xi_j \partial \xi_k} - \frac{\partial \varepsilon_{kn}}{\partial \xi_j \partial \xi_i}\right)$$

$$\frac{\partial \Lambda_{ij}}{\partial \xi_n} - \frac{\partial \Lambda_{in}}{\partial \xi_j} = (x_k - \xi_k)\underbrace{\left(\frac{\partial \varepsilon_{ij}}{\partial \xi_n \partial \xi_k} + \frac{\partial \varepsilon_{kn}}{\partial \xi_j \partial \xi_i} - \frac{\partial \varepsilon_{kj}}{\partial \xi_n \partial \xi_i} - \frac{\partial \varepsilon_{in}}{\partial \xi_j \partial \xi_k}\right)}_{\text{Compatibility condition!}}$$

The integral is a formal result but not very useful

- Usually use a direct integration of components instead

Example: find the 2D displacement field generating the strain

$$\varepsilon_{\alpha\beta} = (3+\nu)a^2\delta_{\alpha\beta} - (1+\nu)(x_\gamma x_\gamma \delta_{\alpha\beta} + 2x_\alpha x_\beta) \qquad \alpha, \beta, \gamma = 1,2$$

① $$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = (3+\nu)a^2 - (1+\nu)(3x_1^2 + x_2^2)$$

$$\Rightarrow \quad u_1 = (3+\nu)a^2 x_1 - (1+\nu)(x_1^3 + x_1 x_2^2) + f(x_2)$$

② $$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = (3+\nu)a^2 - (1+\nu)(3x_2^2 + x_1^2)$$

$$\Rightarrow \quad u_2 = (3+\nu)a^2 x_2 - (1+\nu)(x_2^3 + x_2 x_1^2) + g(x_1)$$

③ $$2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -4(1+\nu)x_1 x_2 = \frac{\partial f}{\partial x_2} + \frac{\partial g}{\partial x_1} - 4(1+\nu)x_1 x_2$$

$$\Rightarrow \quad \frac{\partial f}{\partial x_2} + \frac{\partial g}{\partial x_1} = 0 \qquad \frac{\partial f}{\partial x_2} = -\omega \qquad \frac{\partial g}{\partial x_1} = +\omega$$

$\omega$ = const

$f = C_1 - \omega x_2 \qquad g = C_2 + \omega x_1$

Rigid motion !

Hence $u_\alpha = (3+\nu) a^2 x_\alpha - (1+\nu) x_\gamma x_\gamma x_\alpha$

$+ C_\alpha + \epsilon_{\alpha\beta 3} \omega x_\beta$

Proof that $u_i$ in * generates correct $\varepsilon_{ij}$
will be covered in L11