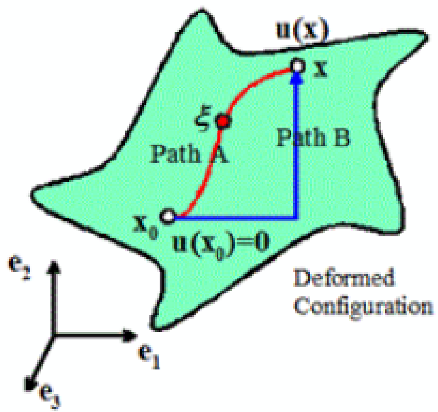
Review: Small strain compatibility conditions

Proposition: if infinitesimal strain field ε_{ij} satisfies $\left(\frac{\partial \varepsilon_{ij}}{\partial \xi_n \partial \xi_k} + \frac{\partial \varepsilon_{kn}}{\partial \xi_j \partial \xi_i} - \frac{\partial \varepsilon_{kj}}{\partial \xi_n \partial \xi_i} - \frac{\partial \varepsilon_{in}}{\partial \xi_j \partial \xi_k}\right) = 0$

Then $u_i(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \Lambda_{ij}(\mathbf{x}, \xi) d\xi_j \qquad \Lambda_{ij} = \varepsilon_{ij}(\xi) + (x_k - \xi_k)\left(\frac{\partial \varepsilon_{ij}}{\partial \xi_k} - \frac{\partial \varepsilon_{kj}}{\partial \xi_i}\right)$ *

satisfies $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$



(Compatibility ensures that integral is path independent in a simply connected solid).

Final problem: show * satisfies

Recall $\frac{d}{dx}\int_a^x f(x,\xi)\,d\xi = f(x,x) + \int_a^x \frac{\partial f}{\partial x}\,d\xi$

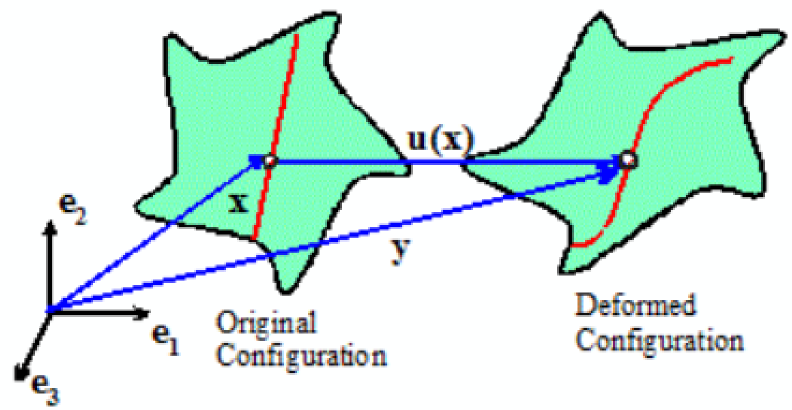
Apply to find $\frac{\partial u_i}{\partial x_n}$

$$\frac{\partial u_i}{\partial x_n} = \varepsilon_{in} + \int_{\underline{x}_0}^{\underline{x}} \frac{\partial \Lambda_{ij}}{\partial x_n}\, d\xi_j$$

$$= \varepsilon_{in} + \int_{\underline{x}_0}^{\underline{x}} \underbrace{\left(\frac{\partial \varepsilon_{ij}}{\partial \xi_n} - \frac{\partial \varepsilon_{nj}}{\partial \xi_i} \right)}_{\text{skew in } i,n} d\xi_j$$

Hence $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_n} + \frac{\partial u_n}{\partial x_i} \right) = \varepsilon_{in}$ ✓

Measures of deformation rate



Particle velocity $\underline{V} = \left.\dfrac{\partial \underline{y}}{\partial t}\right|_{\underline{x}} = \left.\dfrac{\partial \underline{u}}{\partial t}\right|_{\underline{x}}$

$\underline{v}(\underline{x})$ - Lagrangean

$\underline{V}(\underline{y})$ - Eulerean

Velocity Gradient $L = \nabla_y \underline{V}$ $\qquad L_{ij} = \dfrac{\partial v_i}{\partial y_j}$

Properties:

① Quantifies relative velocity of two nearby particles

$$d\underline{V} = \frac{d}{dt}\, d\underline{y} = \underline{v}(\underline{y}+d\underline{y}) - \underline{V}(\underline{y}) = \nabla_y \underline{V}\, d\underline{y}$$

② Note $L = \frac{dF}{dt} F^{-1}$ $\qquad F = \nabla y$

$$d\underline{y} = F d\underline{x} \Rightarrow d\underline{v} = \frac{dF}{dt} d\underline{x}$$

$$d\underline{v} = \underbrace{\frac{dF}{dt} F^{-1}}_{\nabla_y \underline{v} = L} d\underline{y}$$

Stretch Rate and Spin (vorticity) tensor

$D = \mathrm{sym}(L)$ – stretch rate

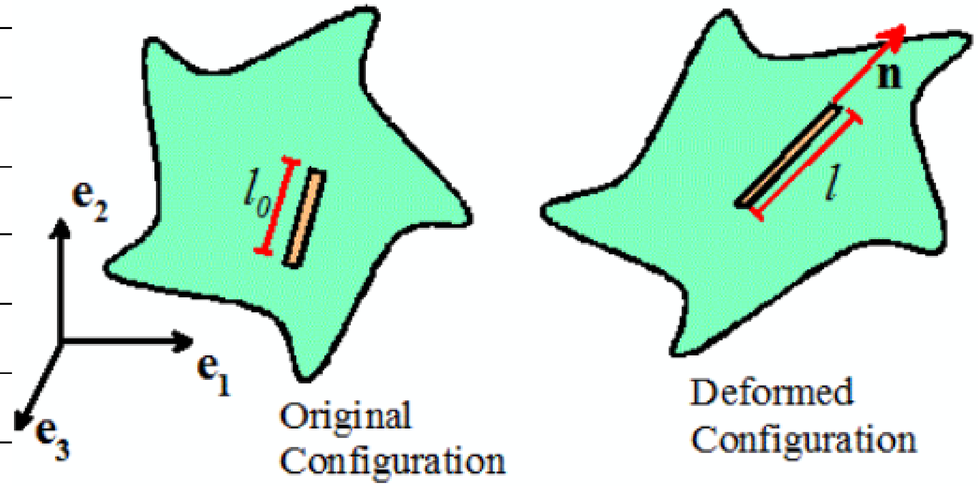
$W = \mathrm{skew}(L)$ – spin

$$D_{ij} = \frac{1}{2}(L_{ij} + L_{ji}) \qquad W_{ij} = \frac{1}{2}(L_{ij} - L_{ji})$$

$$L = D + W$$

Property of D

$$\frac{1}{l}\frac{dl}{dt} = \underline{n}\cdot D\,\underline{n}$$



Proof

$$d\underline{y} = l\,\underline{n} \Rightarrow d\underline{v} = \frac{dl}{dt}\underline{n} + l\frac{d\underline{n}}{dt} = L\underbrace{l\,\underline{n}}_{d\underline{y}}$$

Note $\underline{n}\cdot\underline{n} = 1 \Rightarrow \frac{d\underline{n}}{dt}\cdot\underline{n} = 0$

$$\Rightarrow \frac{dl}{dt} = \underline{n}\cdot L\,l\,\underline{n}$$

$L = D + W$

Recall $\underline{n}\cdot W\underline{n} = 0$

$$\frac{1}{l}\frac{dl}{dt} = \underline{n}\cdot D\underline{n}$$

Vorticity Vector

$$\underline{\omega} = 2 \,\text{dual}\,(W)$$

$$\underline{\omega} = \nabla_y \times \underline{V}$$

Show $\nabla_y \times \underline{V} = 2 \,\text{dual}\,(W)$

Recall $\omega_i = - \epsilon_{ijk} W_{jk}$

$$= - \epsilon_{ijk} \frac{1}{2} \left(\frac{\partial V_j}{\partial y_k} - \frac{\partial V_k}{\partial y_j} \right)$$

$$= \epsilon_{ijk} \frac{\partial V_k}{\partial y_j} \equiv \nabla_y \times \underline{V}$$

Note $W g = \frac{1}{2} \underline{\omega} \times g \quad \forall$ vectors g

Some properties of $\underline{\omega}$

Recall $\left.\dfrac{\partial v_i}{\partial t}\right|_{\underline{x}} = \left.\dfrac{\partial v_i}{\partial t}\right|_{y} + \dfrac{\partial v_i}{\partial y_j} v_j$

$$a_i = \left.\frac{\partial v_i}{\partial t}\right|_{x_k=const} = \left.\frac{\partial v_i}{\partial t}\right|_{y_k=const} + \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + 2W_{ik}v_k$$

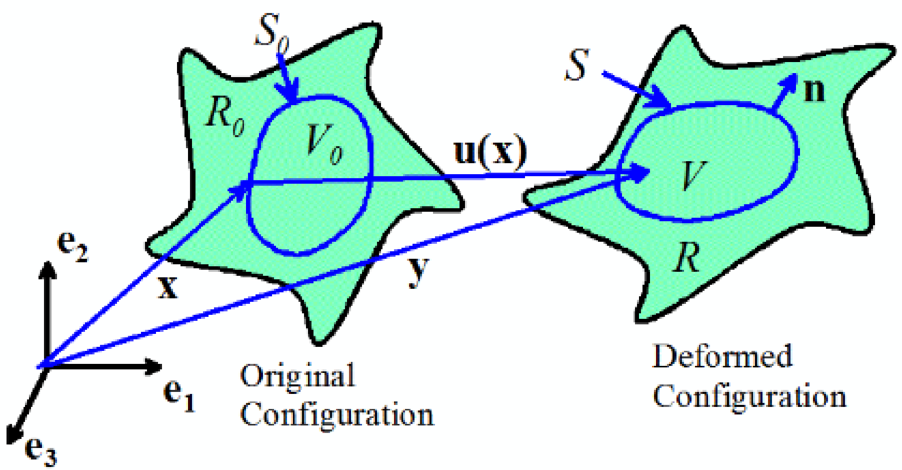
$$a_i = \left.\frac{\partial v_i}{\partial t}\right|_{x_k=const} = \left.\frac{\partial v_i}{\partial t}\right|_{y_k=const} + \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + \in_{ijk}\,\omega_j v_k$$

$$\in_{ijk}\frac{\partial a_k}{\partial y_j} = \left.\frac{\partial \omega_i}{\partial t}\right|_{\mathbf{x}=const} - D_{ij}\omega_j + \frac{\partial v_k}{\partial y_k}\omega_i$$

Rate of change of Volume

Recall $\dfrac{dV}{dV_0} = J = \det(F)$

$$\frac{d}{dt}\left(\frac{dV}{dV_0}\right) = \frac{dJ}{dt} = J\,F^{-T} : \frac{dF}{dt}$$



$$\frac{dJ}{dt} = J\, F^{-1}_{ij} \frac{dF_{ji}}{dt} = J L_{ii} = J D_{ii}$$

To see this note that $\dfrac{dJ}{dt} = \dfrac{dJ}{dF_{ij}} \dfrac{dF_{ij}}{dt}$

Recall $\dfrac{dJ}{dF_{ij}} = J F^{-1}_{ji}$

Some other useful results

① $\dfrac{dE}{dt} = \dfrac{d}{dt}\left(\frac{1}{2}\left(F^T F - I\right)\right) = F^T D F$

② Stretch / rotation – stretch rate / spin relations

Recall $F = RU$ $\quad U = U^T \quad RR^T = I$

Hence $L = \dot{F}F^{-1} = (\dot{R}U + R\dot{U})U^{-1}R^T$

$$= \underbrace{\dot{R}R^T}_{\text{skew}} + R\dot{U}U^{-1}R^T$$

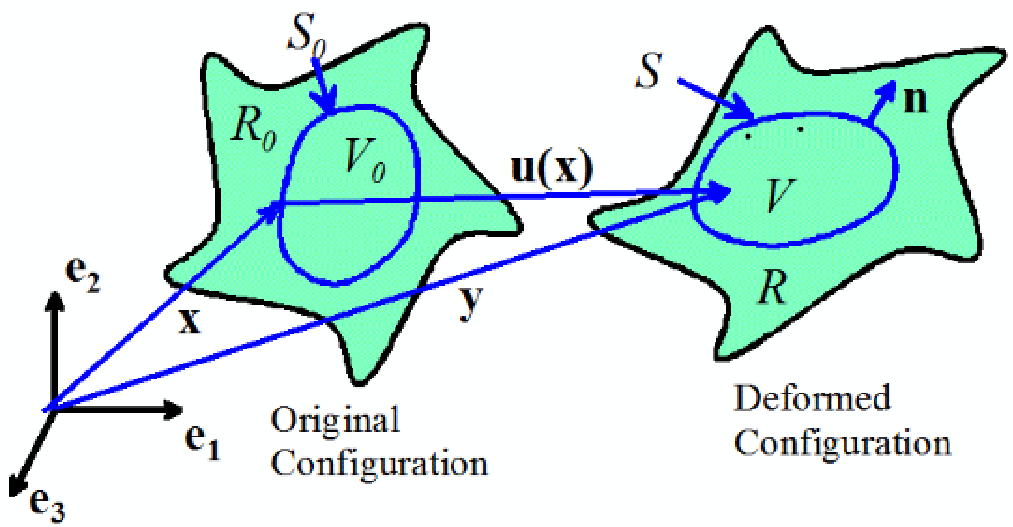
To see $\dot{R}R^T$ is skew note $\frac{d}{dt}(RR^T) = 0$

$$\dot{R}R^T = -R\dot{R}^T = -(\dot{R}R^T)^T$$

Note $D = \text{sym}(R\dot{U}U^{-1}R^T)$

$$W = \dot{R}R^T + \text{skew}(R\dot{U}U^{-1}R^T)$$

Transport relations



V : some material Volume

V_0 : same material vol in ref config

Let $\phi(\underline{x})$ be some scalar material prop (density, KE, temp)

(D_{kk})

Then $$\frac{d}{dt}\int_V \phi \, dV = \int_V \left(\frac{\partial \phi}{\partial t}\Big|_{\underline{x}} + L_{kk}\phi \right) dV$$

$$= \int_V \left(\frac{\partial \phi}{\partial t}\Big|_{y} + \nabla_y \cdot (\phi \underline{v}) \right) dV$$

$$= \int_V \left(\frac{\partial \phi}{\partial t}\Big|_{y} \right) dV + \int_S \phi \underline{v} \cdot \underline{n} \, dA$$