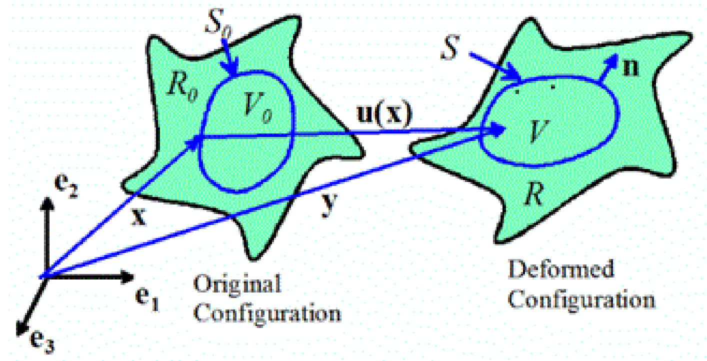
Review

Transport Relations

V Material vol in current config

V_0 Material vol in ref config

$\phi(\mathbf{x})$ Any scalar material property



$$\frac{d}{dt}\int_V \phi\, dV = \int_V \left(\frac{\partial \phi}{\partial t}\bigg|_{\mathbf{x}} + \phi L_{kk} \right) dV = \int_V \left(\frac{\partial \phi}{\partial t}\bigg|_{\mathbf{y}} + \nabla_{\mathbf{y}} \cdot (\phi \mathbf{v}) \right) dV = \int_V \frac{\partial \phi}{\partial t}\bigg|_{\mathbf{y}} dV + \int_S \phi \mathbf{v} \cdot \mathbf{n}\, dA$$

Proof * $\frac{d}{dt}\int_V \phi\, dV = \frac{d}{dt}\int_{V_0} \phi J\, dV_0 = \int_{V_0}\left(\frac{\partial \phi}{\partial t}\Big|_{\underline{x}} J + \phi \frac{dJ}{dt}\right) dV_0$

Recall $\frac{dJ}{dt} = J\, tr(L) = J\, tr(D)$ $\square$

$\Rightarrow * = \int_{V_0}\left(\frac{\partial \phi}{\partial t}\Big|_{\underline{x}} + \phi\, tr(L)\right) J\, dV_0 = \int_V \left(\frac{\partial \phi}{\partial t}\Big|_{\underline{x}} + \phi\, tr(L)\right) dV$

Now recall $\left.\frac{\partial \phi}{\partial t}\right|_{\underline{x}} = \left.\frac{\partial \phi}{\partial t}\right|_{y} + \nabla_y \phi \, \underline{v}$

$$\Rightarrow \square = \left.\frac{\partial \phi}{\partial t}\right|_{y} + \underbrace{(\nabla_y \phi)\underline{v} + \phi \overbrace{\nabla_y \cdot \underline{v}}^{tr(L)}}_{\nabla \cdot (\phi \underline{v})}$$

$L_{ii} = \frac{\partial v_i}{\partial y_i} = \nabla_y \cdot \underline{v}$

Finally note $\int_V \nabla_y \cdot (\phi \underline{v}) \, dV = \int_S \underline{n} \cdot (\phi \underline{v}) \, dA$

(Divergence theorem)

Transport Relations for curves and surfaces

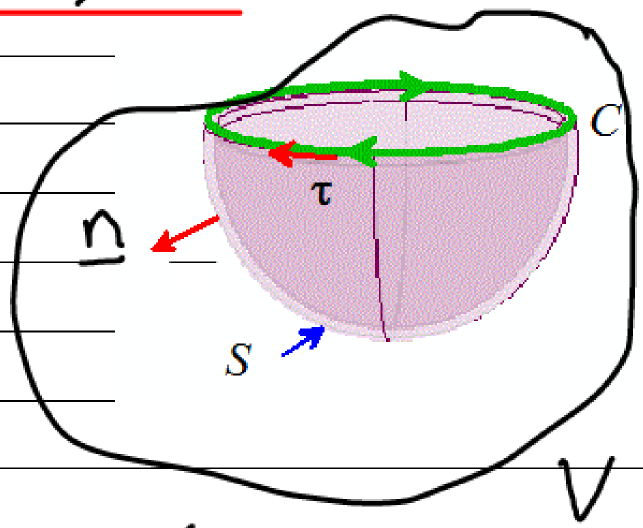
Define C : curve in current config

C_0 : curve in ref config

$\underline{\tau}$: Tangent to curve

S: surface in deformed solid

$\underline{n}$: normal



$$\underbrace{(\underline{\tau}\, ds)}_{\text{material fiber in curr config}} = \underbrace{\underline{F}\, \tau_0\, ds_0}_{\text{Ref config}}$$

$$1. \quad \frac{d}{dt}\int_C \phi \tau_i ds = \int_C \left(\delta_{ij} \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \phi \frac{\partial v_i}{\partial y_j} \right) \tau_j ds$$

$$2. \quad \frac{d}{dt}\int_S \phi n_i dA = \int_S \left(\delta_{ij} \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \delta_{ij}\phi \frac{\partial v_k}{\partial y_k} - \phi \frac{\partial v_j}{\partial y_i} \right) n_j dA$$

Proof:
$$\frac{d}{dt}\int_c \phi\, \underline{\tau}\, ds = \frac{d}{dt}\int_{c_0} \phi\, F\, \underline{\tilde{\tau}}_0\, ds_0$$

$$= \int_{c_0}\left(\frac{\partial \phi}{\partial t}\bigg|_{\underline{x}} \underline{F} + \phi \frac{\partial F}{\partial t} \underbrace{F^{-1} F}_{I}\right) \underline{\tilde{\tau}}_0\, ds_0$$

$$= \int_{c_0}\left(\frac{\partial \phi}{\partial t}\bigg|_{\underline{x}} + \phi \nabla_g \underline{V}\right) F\, \underline{\tilde{\tau}}_0\, ds_0$$

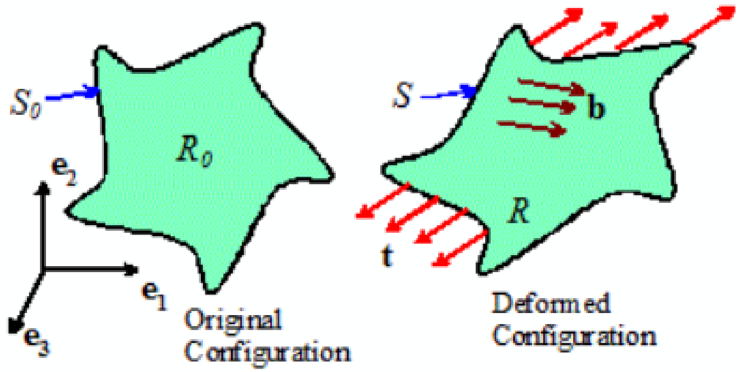
$$= \int_{c}\left(\frac{\partial \phi}{\partial t}\bigg|_{\underline{x}} + \phi \nabla_g \underline{V}\right) \underline{\tau}\, ds$$

After completing ENGN2210 you should

- Be familiar with linear vector spaces relevant to continuum mechanics and able to perform vector and tensor manipulations in Cartesian and curvilinear coordinate systems
- Be able to describe motion, deformation and $\boxed{\text{forces}}$ in a continuum; Up next!
- Be able to derive equations of motion and conservation laws for a continuum ;
- Understand constitutive models for fluids and viscoelastic solids;
- Be able to solve simple boundary value problems for fluids and solids.

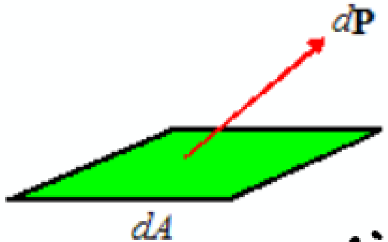
4 Kinetics : Description of internal forces in continua

External Forces



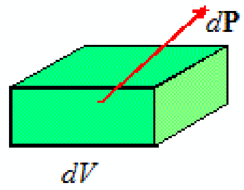
External Traction

$$\underline{t} = \lim_{dA \to 0} \frac{d\underline{P}}{dA}$$



All defined on deformed config

Body force (per unit mass)



$$\underline{b} = \lim_{dV \to 0} \frac{d\underline{P}}{\rho\, dV}$$

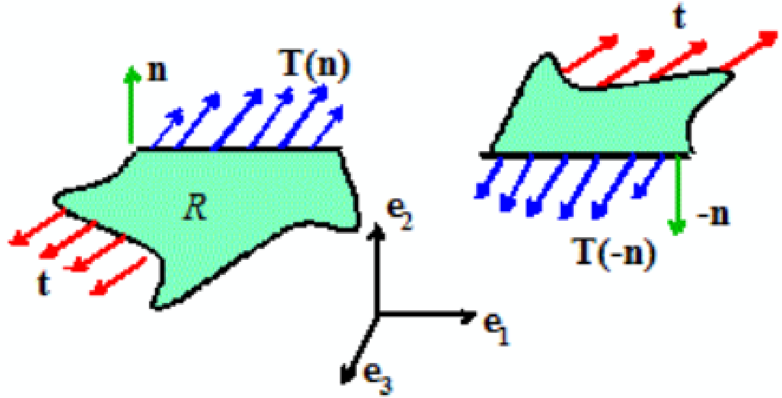
ρ : mass per unit deformed vol

Resultant force

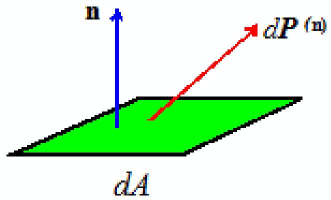
$$\underline{P} = \int_S \underline{t}\, dA + \int_V \rho \underline{b}\, dV$$

Internal Forces

Internal Traction Vector



$$\underline{T}(\underline{n}) = \lim_{dA \to 0} \frac{d\underline{P}(\underline{n})}{dA}$$



Internal traction depends on orientation $\underline{n}$

Cauchy Stress Tensor

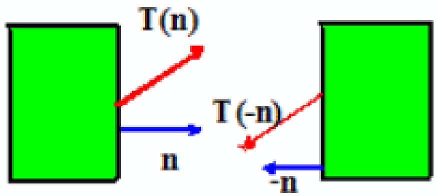
Define stress tensor σ satisfying

$$\underline{T}(\underline{n}) = \underline{n}\,\sigma$$

(note: some authors say $\sigma \underline{n} = \underline{T}(\underline{n})$)

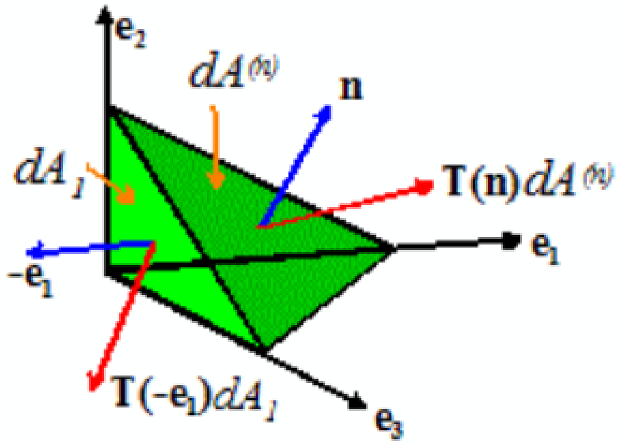
Existence of σ is a consequence of Newtons Laws

① Newton I $\quad \underline{T}(-\underline{n}) = -\underline{T}(\underline{n})$



② Newton III $\quad \underline{F} = m\underline{a}$ for "Cauchy Tetrahedron"

$$\underline{T}(\underline{n})\, dA_n + \sum_{i=1}^{3} \underline{T}(-\underline{e}_i)\, dA_i + \rho \underline{b}\, d\bar{V} = \rho\, d\bar{V}\, \underline{a}$$



Note that $\lim\limits_{dV \to 0} \dfrac{dA_i}{dA_n} = n_i$ (See Appendix E of solidmechanics.org)

$\lim\limits_{dV \to 0} \dfrac{dV}{dA_n} = 0$

$$\Rightarrow \quad \underline{T}(\underline{n}) - \sum_{i=1}^{3} \underline{T}(\underline{e}_i)\, n_i = 0$$

Define $T_j(\underline{e}_i) = \sigma_{ij}$

$$\boxed{T_j(\underline{n}) = n_i \, \sigma_{ij}}$$

Maps a vector onto a vector $\Rightarrow$ σ_{ij} is a tensor