Review

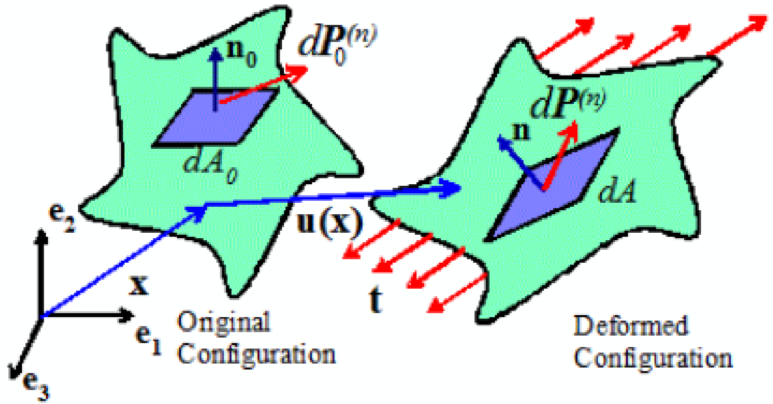
Stress measures

$$\mathbf{F} = \nabla \mathbf{y} \qquad J = \det(\mathbf{F})$$

Cauchy (true) stress $\boldsymbol{\sigma}$

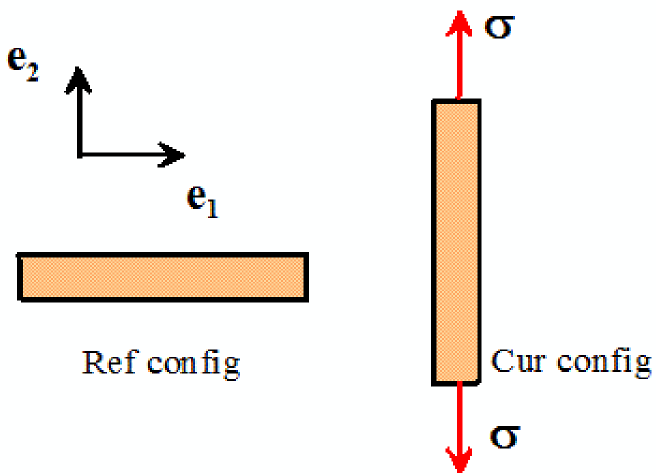
Nominal stress $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$

Material stress $\boldsymbol{\Sigma} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$



Example

Bar is rigid
Calculate nominal and material stress, interpret physically



Here $F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (counterclockwise rotation)

$\underline{\sigma} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}$ $\quad \underline{S} = JF^{-1}\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix} = \underline{\begin{bmatrix} 0 & \sigma \\ 0 & 0 \end{bmatrix}}$

Check: $\overset{n_0}{[1 \quad 0]}\overset{S}{\begin{bmatrix} 0 & \sigma \\ 0 & 0 \end{bmatrix}} = \underset{T(\underline{n})}{[0 \quad \sigma]}$

<u>Material stress</u> $\Sigma = J F^{-1} \sigma F^{-T}$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \sigma & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}$$

Principal Stresses

σ and ε are symmetric

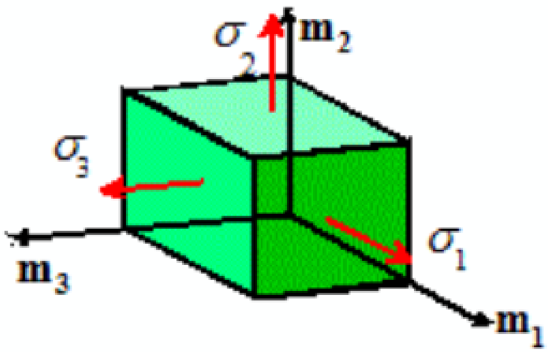
$\Rightarrow$ real eigenvalues $\sigma^{(i)}$ $\leftarrow$ Principal stresses
orthogonal eigenvectors $\underline{m}_i$ $\leftarrow$ " stress directions

$$\Rightarrow \quad \underline{m}_i \, \sigma = \sigma^{(i)} \, \underline{m}_i$$

Physical interpretation : $\exists$ special planes with normal $\underline{m}_i$ on which traction is normal to plane

Can expand stresses

$$\sigma = \sum_{i=1}^{3} \sigma^{(i)} \, \underline{m}_i \otimes \underline{m}_i$$



Stress invariants :

Hydrostatic stress : $p = \frac{1}{3} tr(\sigma) = \frac{1}{3} \sigma_{kk}$

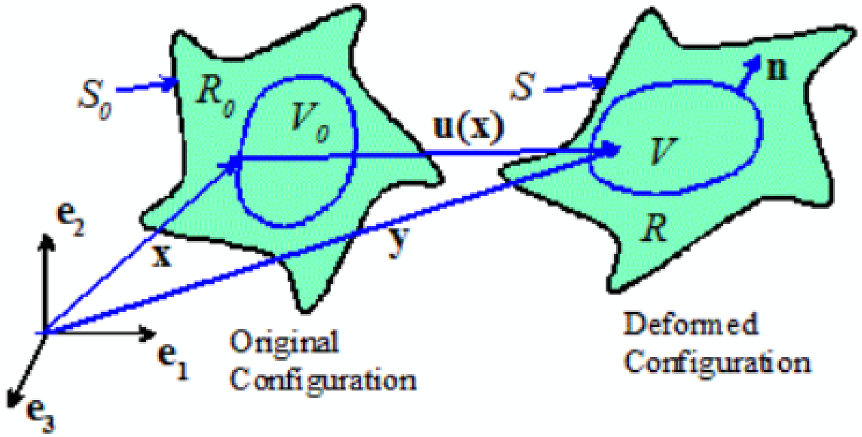
Effective stress : $\sigma_e = \sqrt{\frac{3}{2} \sigma^{DEV} : \sigma^{DEV}}$

$$\sigma^{DEV} = \sigma - pI = \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij}$$

σ^{DEV} : measure of shear stress

4) Conservation Laws

Mass conservation



Let ρ = mass per unit current vol

ρ_0 = " " " ref vol

$$\rho\, dV = \rho J\, dV_0 \Rightarrow \rho_0 = J\rho \qquad J = \det(F)$$

Mass cons: $$\frac{d}{dt}\int_V \rho\, dV = 0 \quad \forall V$$

Transport thm: $$\int_V \left(\left.\frac{\partial \rho}{\partial t}\right|_{\underline{x}} + \rho\, \nabla\cdot\underline{v} \right) dV = 0 \quad \forall V$$

$$\int_V \left(\left.\frac{\partial \rho}{\partial t}\right|_{y} + \nabla\cdot(\rho\underline{v}) \right) dV = 0 \quad \forall V$$

Hence
$$\boxed{\left.\frac{\partial \rho}{\partial t}\right|_{\underset{\sim}{x}} + \rho \nabla\cdot\underline{v} = 0}$$

$$\boxed{\left.\frac{\partial \rho}{\partial t}\right|_{\underset{\sim}{y}} + \nabla\cdot(\rho\underline{v}) = 0}$$

Linear Momentum

$$\underline{F} = \frac{d}{dt}(m\underline{v})$$

$$\int_S \underline{n}\,\sigma\, dA + \int_V \rho\,\underline{b}\, dV = \frac{d}{dt}\int_V \rho\,\underline{v}\, dV$$

Div thm ... Transport thm

$$\int_V \nabla_y\cdot\sigma\, dV + \int_V \rho\underline{b}\,dV = \int_V \underbrace{\left(\left.\frac{\partial \rho\underline{v}}{\partial t}\right|_{\underset{\sim}{x}} + \rho\underline{v}\,\nabla\cdot\underline{v}\right)}_{\square} dV$$

$$\square = \int_V \rho\left.\frac{\partial \underline{v}}{\partial t}\right|_{\underset{\sim}{x}} + \underline{v}\left.\frac{\partial \rho}{\partial t}\right|_{\underset{\sim}{x}} + \rho\underline{v}\,\nabla\cdot\underline{v}$$

Mass conservation $\Rightarrow$ $\underline{v} \left.\frac{\partial \rho}{\partial t}\right|_{\underline{x}} = -\underline{v}\, \rho\, \nabla \cdot \underline{v}$

Collect all $\Rightarrow$ $\int_V \left(\nabla_y \cdot \sigma + \rho \underline{b} - \rho \left.\frac{\partial \underline{v}}{\partial t}\right|_{\underline{x}} \right) dV = 0 \quad \forall V$

$$\boxed{\nabla_y \cdot \sigma + \rho \underline{b} = \rho \left.\frac{\partial \underline{v}}{\partial t}\right|_{\underline{x}}}$$

$\underline{v} = \underline{0}$ for statics

$$\boxed{\frac{\partial \sigma_{ij}}{\partial y_i} + \rho b_j = \rho \left.\frac{\partial v_j}{\partial t}\right|_{\underline{x}}}$$

Angular momentum $\quad y \times \underline{F} = \frac{d}{dt}(y \times m\underline{v})$

Continuum version

$$\int_S \underset{\sim}{y} \times \underline{n}\,\sigma \, dA + \int_V \underset{\sim}{y} \times \rho \underline{b}\, dV = \frac{d}{dt}\int_V \underset{\sim}{y} \times \rho \underline{v}\, dV$$

① ② ③

$$① \Rightarrow \int_S \epsilon_{ijk}\, y_j\, n_\ell\, \sigma_{\ell k}\, dA = \int_V \frac{\partial}{\partial y_\ell}\left(\epsilon_{ijk}\, y_j\, \sigma_{\ell k}\right) dV$$

$$= \int_V \Big(\epsilon_{ijk}\,\sigma_{jk} + \underbrace{\epsilon_{ijk}\, y_j \frac{\partial \sigma_{\ell k}}{\partial y_\ell}}_{y \times \nabla_y \cdot \sigma}\Big)\, dV \quad ④$$

$$③ = \int_V \left(\frac{\partial}{\partial t}(\underset{\sim}{y} \times \rho \underline{v})\Big|_{\underline{x}} + \underset{\sim}{y} \times \rho \underline{v}\; \nabla \cdot \underline{v} \right) dV$$

(Transport thm)

$$= \int_V \left(\frac{\partial \underline{y}}{\partial t} \times \cancel{\rho} \underline{v} + \underline{y} \times \rho \frac{\partial \underline{v}}{\partial t}\bigg|_{\underline{x}} + \underbrace{\underline{y} \times \underline{v} \frac{\partial \rho}{\partial t} + \underline{y} \times \rho \underline{v}\, \nabla \cdot \underline{v}}_{\text{cancel (mass conserv)}} \right) dV$$

$=0$

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$$4 + 2 - 5 = 0$$

$$\int_V \epsilon_{ijk} \sigma_{jk}\, dV + \int_V \underline{y} \times \underbrace{\left(\nabla_y \sigma + \rho \underline{b} - \rho \frac{\partial \underline{v}}{\partial t}\bigg|_{\underline{x}} \right)}_{=0 \text{ (BLM)}} dV = 0$$

$=0 \ \forall V$

$$\epsilon_{ijk} \sigma_{jk} = 0 \Rightarrow \epsilon_{ipq} \epsilon_{ijk} \sigma_{jk} = 0$$

$$\Rightarrow (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}) \sigma_{jk} = 0$$

$$\boxed{\sigma_{pq} = \sigma_{qp}} \Rightarrow \boxed{\sigma = \sigma^T}!$$