Review

Conservation Laws

Mass
\[ \frac{\partial \rho}{\partial t}_{x=\text{const}} + \rho \frac{\partial v_i}{\partial y_i} = 0 \]
\[ \frac{\partial \rho}{\partial t}_{y=\text{const}} + \frac{\partial \rho v_i}{\partial y_i} = 0 \]

Linear Momentum
\[ \frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho a_i \]

Angular Momentum
\[ \sigma_{mn} - \sigma_{nm} = 0 \]

Conservation Laws in terms of other stress measures

Define:  
\[ F = \nabla y \]  
\[ \varepsilon = \int \sigma \]  
\[ S = \int F^{-T} \sigma F^{-T} \]  
\[ \varepsilon = \int F^{-T} \sigma F^{-T} \]
Then \textbf{BLM}:

\[ \nabla \cdot S^T + p_0 b = p_0 \frac{dr}{dt} \bigg|_x \]

\[ \frac{\partial S_{ij}}{\partial x_i} + p_0 b_j = p_0 \frac{\partial v_j}{\partial t} \bigg|_x \]

\[ \nabla \cdot (F \xi) + p_0 b = p_0 \frac{\partial v_j}{\partial t} \bigg|_x \]

\[ \frac{\partial}{\partial x_i} \sum_{k} F_{jk} + p_0 b_j = p_0 \frac{\partial v_j}{\partial t} \bigg|_x \]

Notes: Divergences are \textit{wrt} \(x\) instead of \(y\)

Other authors may use different convention for \(D\).
BAM: \( \sigma = \sigma^T \quad FS = (FS)^T = S^T F^T \)

\[ \varepsilon = \varepsilon^T \]

**Special case: Infinitesimal Deformations**

Let \( F = (I + D\varepsilon) \quad D\varepsilon; \quad D\varepsilon \ll 1 \)

We approximate \( \sigma = S = \varepsilon \)

Approximate BAM as \( \frac{\partial \sigma_{ij}}{\partial x_i} + p_0 b_j = p_0 \frac{\partial \varepsilon_{ij}}{\partial t} \)
Mechanical Work and Energy

\[ F \cdot v = \frac{d}{dt} \left( \frac{1}{2} m v^2 \right) \]

For a continuum

\[ f_p = \int_S (n \sigma) \cdot v \, dA + \int_V p b \cdot v \, dV = \int_V \sigma \cdot \dot{V} \, dV + \frac{d}{dt} \int_V \rho v^2 \, dV \]

\[ \text{Stress power} \quad \text{Stored or dissipated} \]

Proof:

\[ \sigma = \int_S n_i \sigma_{ij} v_j \, dA = \int_V \frac{\partial}{\partial y_i} (\sigma_{ij} v_j) \, dV \]

\[ = \int_V \left( \sigma_{ij} \frac{\partial v_j}{\partial y_i} + v_j \frac{\partial \sigma_{ij}}{\partial y_i} \right) \, dV \]
\[ \Theta = \int_V v_j \left( \frac{\partial \phi_j}{\partial y_i} + \rho b_j \right) \, dv = \int_V v_j \rho \frac{\partial v_i}{\partial t} \, dv \]

\[ = \int_{V_0} v_j \rho \frac{\partial v_j}{\partial t} \, dV_0 = \int_{V_0} \frac{d}{dt} \left( \frac{1}{2} \rho_0 (v_1)^2 \right) \, dV_0 \]

\[ = \frac{d}{dt} \int_{V_0} \frac{1}{2} \rho (v_1)^2 \, dV \]

\[ \Rightarrow \rho = \Theta + \Delta = \int_V \sigma \cdot \omega \, dV + \frac{d}{dt} \int_V \frac{1}{2} \rho (v_1)^2 \, dV \]

\[ \text{KE} \]
Work-energy in terms of other stress measures

\[ r_p = \int_{V_0} \mathbf{e} : \mathbf{D} \, dV_0 + \frac{d}{dt} (KE) \]

\[ r_p = \int_{V_0} \mathbf{S} \cdot \frac{d\mathbf{F}}{dt} \, dV_0 + \frac{d}{dt} (KE) \]

\[ r_p = \int_{V_0} \mathbf{e} : \frac{d\mathbf{E}}{dt} \, dV_0 + \frac{d}{dt} (KE) \]

**Proof:**

1. \[ r_p = \int_{V} \mathbf{e} : \mathbf{D} \, dV = \int_{V_0} \mathbf{e} : \mathbf{D} \, J \, dV_0 - \int_{V_0} \mathbf{e} : \partial \mathbf{D} \, dV_0 \]
2. Recall \[ L = \frac{dF}{dt} F^{-1} \] (see kinematic notes)

\[ \mathbf{r}_\mathbf{p} = \int_V \mathbf{O}_{ij} \mathbf{K}_{ii} \ d\mathbf{V} = \int_{V_0} \mathbf{O}_{ij} \ \frac{dF_j}{dt} \ F_i^{-1} \ J dV_0 \]

\[ = \int_{V_0} J \ F_i \mathbf{O}_{ij} \ \frac{dF_j}{dt} \ dV_0 \]

\[ = \int_{V_0} S_{ij} \mathbf{O}_{ij} \ \frac{dF_j}{dt} \ dV_0 \]

3. Recall \[ J = F^{-T} \frac{dE}{dt} F^{-1} \] (HW #3)

\[ \Rightarrow \int_{V_0} \mathbf{\sigma} : \mathbf{D} \ J dV_0 = \int_{V_0} J \mathbf{\sigma} : \left( F^{-T} \frac{dE}{dt} F^{-1} \right) dV_0 \]
\[ p = \int_{V_0} \varepsilon \cdot dE \ dV_0 + \frac{d}{dt} (KE) \]

Note: \( \varepsilon : D \) \( \Rightarrow \) rate of work done by stresses per unit ref volume
\( S \cdot \dot{\varepsilon} \) \( \Rightarrow \) Work-conjugate pair
3. The figure shows a test designed to measure the viscosity of a fluid. The sample is a hollow cylinder with internal radius \( a_0 \) and external radius \( a_1 \). The inside diameter is bonded to a fixed rigid cylinder. The external diameter is bonded inside a rigid tube, which is rotated with angular velocity \( \omega(t) \). Assume that all material particles in the specimen (green) move circumferentially, with a velocity field (in spatial coordinates) \( \mathbf{v} = v_\theta(r,t) \mathbf{e}_\theta \).

(a) Calculate the spatial velocity gradient \( \mathbf{L} \) in the basis \( \{ \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z \} \) and hence deduce the stretch rate tensor \( \mathbf{D} \).

\[
\mathbf{L} = \nabla_y \mathbf{v} = v_\theta(r) \mathbf{e}_\theta \otimes \left( \frac{\partial}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{\partial}{\partial z} \mathbf{e}_z \right)
\]

\[
= \frac{\partial}{\partial r} v_\theta \mathbf{e}_\theta \otimes \mathbf{e}_r - \frac{v_\theta}{r} \mathbf{e}_r \otimes \mathbf{e}_\theta
\]

\[
\mathbf{D} = \text{sym}(\mathbf{L}) = \frac{1}{2} \left( \frac{\partial}{\partial r} - \frac{v_\theta}{r} \right) \left( \mathbf{e}_\theta \otimes \mathbf{e}_r + \mathbf{e}_r \otimes \mathbf{e}_\theta \right)
\]
(b) Calculate the acceleration field

\[ \frac{\partial \mathbf{u}}{\partial t} \bigg|_y = \frac{\partial \mathbf{u}}{\partial t} \bigg|_y + \mathbf{L} \cdot \mathbf{u} = -\frac{V_0^2}{r} \mathbf{e}_r \]

(c) Suppose that the specimen is homogeneous, has mass density \( \rho \), and may be idealized as a viscous fluid, in which the Kirchhoff stress is related to stretch rate by

\[ \mathbf{\tau} = 2\mu \mathbf{D} + p(r,t)\mathbf{I} \]

where \( p \) is a hydrostatic pressure (to be determined) and \( \mu \) is the viscosity. Use this to write down an expression for the Cauchy stress tensor in terms of \( p \), expressing your answer as components in \( \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\} \)

Note: \( \mathbf{J} = 1 \)

\[ \mathbf{\sigma} = \mathbf{\tau} = \mu \left( \frac{\partial \mathbf{u}_\theta}{\partial r} - \frac{\partial \mathbf{u}_r}{\partial \theta} \right) \left( \mathbf{e}_\theta \otimes \mathbf{e}_r + \mathbf{e}_r \otimes \mathbf{e}_\theta \right) + \mathbf{p} \mathbf{I} \]
(d) Assume steady deformation. Express the equations of \( v_\theta(r, t) \) in terms of \( v_\theta(r, t) \).

\[
\nabla_y \cdot \sigma = \rho \alpha
\]

\[
\sigma \cdot \left( \frac{\partial}{\partial r} \frac{e_r}{r} + \frac{1}{r} \frac{\partial}{\partial \theta} e_\theta + \frac{\partial}{\partial z} e_z \right) = -\rho \frac{v_\theta^2}{r}
\]

\[
+ \mu \frac{\partial}{\partial r} \left( \frac{\partial v_\theta}{\partial r} \right) e_\theta + 2\mu \left( \frac{\partial v_\theta}{\partial r} \right) e_\theta
\]

\[
+ \frac{\partial}{\partial z} e_r = -\frac{v_\theta^2}{r} e_r
\]

(e) Solve the equilibrium equation, together with appropriate boundary conditions, to calculate \( v_\theta(r, t) \) and \( p(r) \). (The pressure can only be determined to within an arbitrary constant).

\[
Solve \quad \frac{\partial}{\partial r} \left( \frac{\partial v_\theta}{\partial r} \right) + 2 \left( \frac{\partial v_\theta}{\partial r} \right) = 0
\]
Boundary condition: \( \nu_0 = 0 \quad r = a_0 \)
\( \nu_0 = \omega q_1 \quad r = a_1 \)

\[
\frac{d\rho}{dr} + \frac{\nu_0^2}{r} \quad \text{solve within a constant}
\]

\[
c1 := (\text{diff}(\nu q(r), r) - \nu q(r)/r): \quad \frac{d\nu}{dr} = \frac{\nu}{r}
\]
\[
diffeq1 := \text{diff}(c1, r) + 2*c1/r=0:
\]
\[
bc := \nu q(a0)=0, \nu q(a1)=a1*w: \quad \text{Boundary Conditions}
\]
\[
vqsol := \text{solve}(\text{ode}([\text{diffeq1}, bc], \nu q(r)), \text{IgnoreSpecialCases}))\[1\]
\[
\frac{a1^2 \omega (a0^2 - r^2)}{r (a0^2 - a1^2)}
\]
\[
\nu_0 = \frac{a1^2 \omega}{a_i^2 - a_o^2} \left( r - \frac{a_o^2}{r} \right)
\]
\[
diffeq2 := \text{diff}(p(r), r) = -\rho o * vqsol^2/r:
\]
\[
psol := \text{solve}(\text{ode}([\text{diffeq2}], p(r)), \text{IgnoreSpecialCases}))\[1\]
\[
C16 + \frac{a1^4 \rho o w^2 (a0^4 - r^4 + 4 a0^2 r^2 \ln(r))}{2 r^2 (a0^2 - a1^2)^2}
\]
\[
p = \text{const of integration}
\]