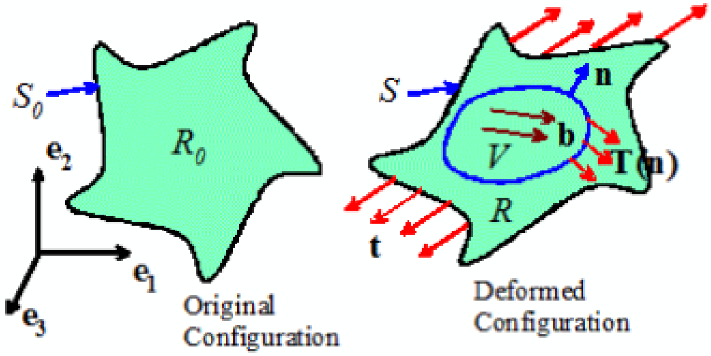
Review

Conservation Laws

Mass
$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho \frac{\partial v_i}{\partial y_i} = 0$$

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0$$



Mass density $\rho_0 = J\rho$ ρ

Linear Momentum $\dfrac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho a_i$

Angular Momentum $\sigma_{mn} - \sigma_{nm} = 0$

Conservation Laws in terms of other stress measures

Define: $F = \nabla y$, $\tau = J\sigma$ $S = JF^{-1}\sigma$ $\Sigma = JF^{-1}\sigma F^{-T}$

Then BLM:

$$\nabla \cdot S^T + \rho_0 \underline{b} = \rho_0 \left.\frac{\partial \underline{v}}{\partial t}\right|_{\underset{\sim}{x}}$$

$$\frac{\partial S_{ij}}{\partial x_i} + \rho_0 b_j = \rho_0 \left.\frac{\partial v_j}{\partial t}\right|_{\underline{x}}$$

$$\nabla \cdot (F \Sigma) + \rho_0 \underline{b} = \rho_0 \left.\frac{\partial \underline{v}}{\partial t}\right|_{\underset{\sim}{x}}$$

$$\frac{\partial}{\partial x_i} \Sigma_{ik} F_{jk} + \rho_0 b_j = \rho_0 \left.\frac{\partial v_j}{\partial t}\right|_{\underline{x}}$$

Notes: Divergences are wrt $\underline{x}$ instead of y

Other authors may use different conventions for $\nabla \cdot \underset{\sim}{S}$

BAM: $\sigma = \sigma^T \qquad FS = (FS)^T = S^T F^T$

$$\Sigma = \Sigma^T$$

Special case: Infinitesimal Deformations

Let $F = (I + \nabla \underline{u}) \qquad \nabla \underline{u} : \nabla \underline{u} \ll 1$

We approximate $\sigma \simeq S \simeq \Sigma$

Approximate BLM as $\dfrac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 b_j = \rho_0 \dfrac{\partial v_j}{\partial t}\Big|_{\underline{x}}$

Mechanical Work and Energy

$$\left(\underline{F} \cdot \underline{v} = \frac{d}{dt}\left(\frac{1}{2} m |\underline{v}|^2\right) \right)$$

For a continuum

$$r_p = \underbrace{\int_S (n\sigma)\cdot \underline{v}\, dA}_{①} + \underbrace{\int_V \rho\, \underline{b}\cdot\underline{v}\, dV}_{②} = \underbrace{\int_V \underbrace{\sigma \cdot\cdot L}_{\sigma : D}\, dV}_{} + \frac{d}{dt}\underbrace{\int_V \frac{1}{2}\rho |\underline{v}|^2 dV}_{KE}$$

Stress power

Stored or dissipated

Proof:

$$① = \int_S n_i \sigma_{ij} v_j\, dA = \int_V \frac{\partial}{\partial y_i}(\sigma_{ij} v_j)\, dV \qquad ③$$

$$= \int_V \left(\underbrace{\sigma_{ij} \frac{\partial v_j}{\partial y_i}}_{⑤} + \underbrace{v_j \frac{\partial \sigma_{ij}}{\partial y_i}}_{③} \right) dV$$

$$③ + ② = \int_V v_j \left(\frac{\partial \sigma_{ij}}{\partial y_i} + \rho b_j \right) dV = \int_V v_j \rho \left. \frac{\partial v_j}{\partial t} \right|_{\underline{x}} dV$$

$$= \int_{V_0} v_j \rho \frac{\partial v_j}{\partial t} J \, dV_0 = \int_{V_0} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 |\underline{v}|^2 \right) dV_0$$

$$= \frac{d}{dt} \int_V \frac{1}{2} \rho |\underline{v}|^2 dV \qquad ④$$

$$\Rightarrow r_p = ④ + ⑤ = \int_V \sigma \cdot\cdot L \, dV + \underbrace{\frac{d}{dt} \int_V \frac{1}{2} \rho |\underline{v}|^2 dV}_{KE}$$

Work - energy in terms of other stress measures

① $$r_p = \int_{V_0} \tau : D \, dV_0 + \frac{d}{dt}(KE)$$

② $$r_p = \int_{V_0} S \cdot\cdot \frac{dF}{dt} \, dV_0 + \frac{d \, KE}{dt}$$

③ $$r_p = \int_{V_0} \Sigma : \frac{dE}{dt} \, dV_0 + \frac{d \, KE}{dt}$$

Proof: ① : $r_p = \int_V \sigma : D \, dV = \int_{V_0} \sigma : D \, J \, dV_0 = \int_{V_0} \tau : D \, dV$

(2) Recall $L = \frac{dF}{dt} F^{-1}$ (see kinematics notes)

$$r_p = \int_V \sigma_{ij} L_{ji} \, dV = \int_{V_0} \sigma_{ij} \frac{dF_{jk}}{dt} F_{ki}^{-1} J \, dV_0$$

$$= \int_{V_0} \underbrace{J F_{ki}^{-1} \sigma_{ij}}_{S_{kj}} \frac{dF_{jk}}{dt} \, dV_0$$

(3) Recall $D = F^{-T} \frac{dE}{dt} F^{-1}$ (HW #3)

$$\Rightarrow \quad \int_{V_0} \sigma : D \, J \, dV_0 = \int_{V_0} J \sigma : \left(F^{-T} \frac{dE}{dt} F^{-1} \right) dV_0$$

$$\Gamma_p = \int_{V_0} \Sigma : \frac{dE}{dt}\, dV_0 + \frac{d}{dt}(KE)$$

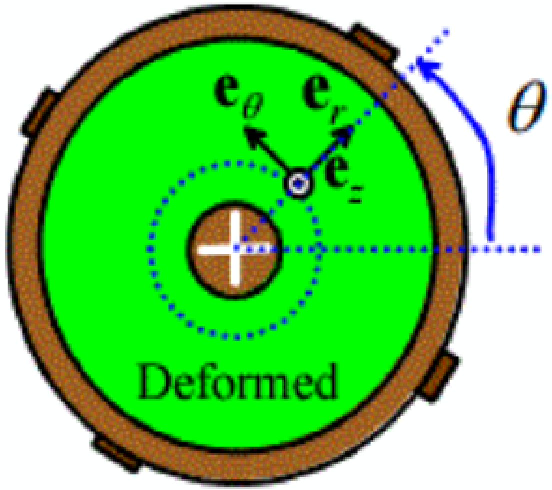
Note:

$$\left.\begin{array}{l} \tau : D \\ S : \dot{F} \\ \Sigma : \dot{E} \end{array}\right\}$$

rate of work done by stresses per unit ref volume

Work-conjugate pairs

3. The figure shows a test designed to measure the viscosity of a fluid. The sample is a hollow cylinder with internal radius a_0 and external radius a_1. The inside diameter is bonded to a fixed rigid cylinder. The external diameter is bonded inside a rigid tube, which is rotated with angular velocity $\omega(t)$. Assume that all material particles in the specimen (green) move circumferentially, with a velocity field (in spatial coordinates) $\mathbf{v} = v_\theta(r,t)\mathbf{e}_\theta$.



(a) Calculate the spatial velocity gradient $\mathbf{L}$ in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and hence deduce the stretch rate tensor $\mathbf{D}$.

$$L = \nabla_y \underline{v} = v_\theta(r)\underline{e_\theta} \otimes \left(\frac{\partial}{\partial r}\underline{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\underline{e_\theta} + \frac{\partial}{\partial z}\underline{e_z} \right)$$

$$= \frac{\partial v_\theta}{\partial r}\underline{e_\theta} \otimes \underline{e_r} - \frac{v_\theta}{r}\underline{e_r} \otimes \underline{e_\theta}$$

$$D = sym(L) = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)\left[\underline{e_\theta} \otimes \underline{e_r} + \underline{e_r} \otimes \underline{e_\theta}\right]$$

(b) Calculate the acceleration field

$$\left.\frac{\partial \underline{v}}{\partial t}\right|_{\underline{x}} = \left.\frac{\partial \underline{v}}{\partial t}\right|_{\underline{y}} + L\underline{v} = -\frac{v_\theta^2}{r}\,\underline{e_r}$$

(c) Suppose that the specimen is homogeneous, has mass density ρ, and may be idealized as a viscous fluid, in which the Kirchhoff stress is related to stretch rate by

$$\boldsymbol{\tau} = 2\mu\mathbf{D} + p(r,t)\mathbf{I}$$

where p is a hydrostatic pressure (to be determined) and μ is the viscosity. Use this to write down an expression for the Cauchy stress tensor in terms of p, expressing your answer as components in $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$

Note $J = 1$

$$\sigma = \tau = \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)\left(\underline{e_\theta} \otimes \underline{e_r} + \underline{e_r} \otimes \underline{e_\theta}\right) + p\underline{I}$$

(d) Assume steady deformation. Express the equations of LMB in terms of $v_\theta(r,t)$.

$$\nabla_y \cdot \sigma = \rho \underline{a}$$

$$\sigma \cdot \left(\frac{\partial}{\partial r}\underline{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\underline{e_\theta} + \frac{\partial}{\partial z}\underline{e_z} \right) = -\rho \frac{v_\theta^2}{r}$$

$$+ \mu \frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)\underline{e_\theta} + 2\mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)\underline{e_\theta}$$

$$+ \frac{\partial p}{\partial r}\underline{e_r} = -\frac{v_\theta^2}{r}\underline{e_r}$$

(e) Solve the equilibrium equation, together with appropriate boundary conditions, to calculate $v_\theta(r,t)$, and $p(r)$. (The pressure can only be determined to within an arbitrary constant).

Solve $\frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) + \frac{2}{r}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) = 0$

Boundary conditions : $v_\theta = 0 \quad r = a_0$
$v_\theta = \omega a_1 \quad r = a_1$

$\frac{\partial p}{\partial r} = -\frac{v_\theta^2}{r}$ — solve within a constant

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c1 := (diff(vq(r),r) - vq(r)/r):
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— $\partial v/\partial r - v/r$

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diffeq1 := diff(c1,r) + 2*c1/r=0:
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—

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bc := vq(a0)=0,vq(a1)=a1*w:
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— Boundary Conditions

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vqsol := simplify(solve(ode({diffeq1,bc}, vq(r)),IgnoreSpecialCases))[1]
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$$\frac{a1^2 w (a0^2 - r^2)}{r (a0^2 - a1^2)}$$

$v_\theta = \frac{a_1^2 \omega}{a_1^2 - a_0^2}\left(r - \frac{a_0^2}{r}\right)$

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diffeq2 := diff(p(r),r) = -rho*vqsol^2/r:
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$\partial P/\partial r = -\rho v^2/r$

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psol := simplify(solve(ode({diffeq2}, p(r)),IgnoreSpecialCases))[1]
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$$C16 + \frac{a1^4 \, rho \, w^2 (a0^4 - r^4 + 4\, a0^2 r^2 \ln(r))}{2 r^2 (a0^2 - a1^2)^2}$$

const of integration