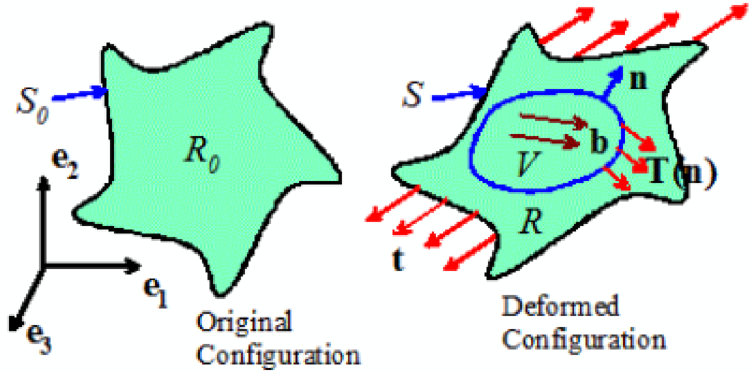
Review

Principle of Virtual Work



Let $\boldsymbol{\sigma}$ be a symmetric Cauchy stress field

1. Stress satisfies BLM and BCs

$$\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho a_i \qquad n_i \sigma_{ij} = t_j$$

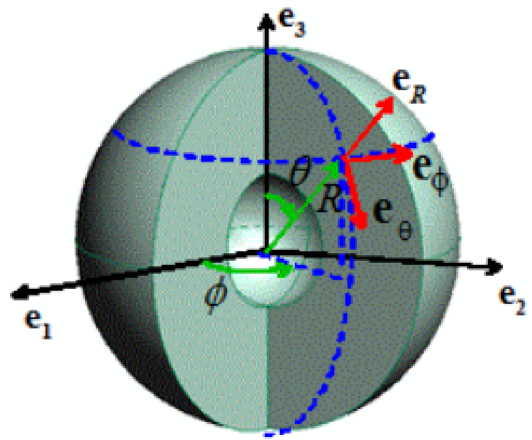
2. Admissible velocity field $\delta v_i(\mathbf{y})$ $\quad \delta D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right)$

3. Virtual Work $\quad \delta P = \int_V \sigma_{ij} \delta D_{ij} dV - \int_V b_j \delta v_j dV + \int_V \left.\frac{\partial v_j}{\partial t}\right|_{\mathbf{x}} \delta v_j dV - \int_S t_j \delta v_j dA$

PVW: If stress satisfies (1),

If $\delta P = 0$ for all $\delta v_i(\mathbf{y})$ stress satisfies (1)

PVW can be used to derive BLM for systems with reduced DOF



Example : Derive BLM equations for spherically symmetric problems

Assumptions $\quad \underline{v} = v(R)\,\underline{e}_R$

$$\underline{b} = b(R)\,\underline{e}_R$$

$$\underline{t} = \begin{cases} t_R(b)\,\underline{e}_R & R = b \\ t_R(a)\,\underline{e}_R & R = a \end{cases}$$

Cauchy stress : σ : $\begin{bmatrix} \sigma_{RR} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{bmatrix}$

To derive BLM assume $\delta \underline{v} = \delta v(R)\,\underline{e}_R$

Virtual Velocity Gradient $\nabla_y \delta \underline{v} = L$

$$L = \delta v(R)\, \underline{e_R} \otimes \left(\frac{\partial}{\partial R} \underline{e_R} + \frac{1}{R}\frac{\partial}{\partial \theta} \underline{e_\theta} + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi} \underline{e_\phi} \right)$$

$$= \frac{\partial \delta v}{\partial R} \underline{e_R} \otimes \underline{e_R} + \frac{\delta v}{R} \underline{e_\theta} \otimes \underline{e_\theta} + \frac{\delta v}{R} \underline{e_\phi} \otimes \underline{e_\phi}$$

Subst into PVW: $\delta P =$

$$\int_a^b \left(\sigma_{RR} \frac{\partial \delta v}{\partial R} + (\sigma_{\theta\theta} + \sigma_{\phi\phi}) \frac{\delta v}{R} \right) 4\pi R^2 dR - \int_a^b b(R)\, \delta v\, 4\pi R^2 dR$$

$$+ \int_a^b \rho \left.\frac{\partial v}{\partial t}\right|_{\underline{x}} \delta v\, 4\pi R^2 dR$$

$$- 4\pi b^2 \delta v(b)\, t_r(b) - 4\pi a^2 \delta v(a)\, t_r(a) = 0$$

for all δv

Need to integrate terms with $\frac{\partial \delta u}{\partial R}$ by parts

$$\int_a^b \sigma_{RR} R^2 \frac{\partial \delta u}{\partial R} = \int_a^b \left(\frac{\partial}{\partial R}\left(\sigma_{RR} \delta u R^2\right) - \delta u \frac{\partial(\sigma_{RR} R^2)}{\partial R} \right) dR$$

$$b^2 \delta u(b) \sigma_{RR} - a^2 \delta u(a) \sigma_{RR} - \int_a^b \delta u \left(\frac{\partial \sigma_{RR}}{\partial R} + \frac{2\sigma_{RR}}{R} \right) R^2 dR$$

Collect terms

$$-\int_a^b \left(\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + b - \rho \frac{\partial v_R}{\partial t} \right) \delta u \, 4\pi R^2 dR$$

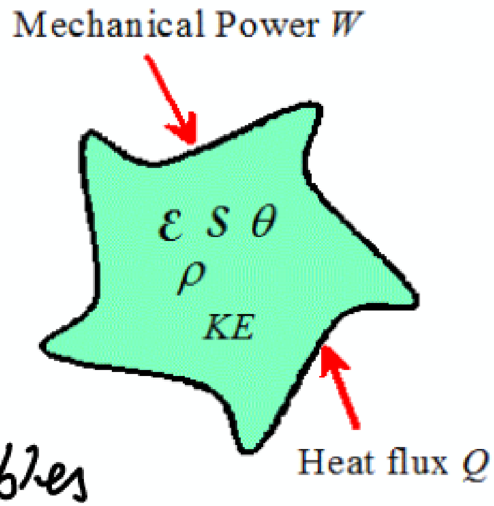
$$+ 4\pi b^2 \delta u(b) \left(\sigma_{RR} - t_R(b) \right) - 4\pi a^2 \delta u(a) \left(\sigma_{RR} + t_R(a) \right)$$

$$= 0 \quad \forall \, \delta u$$

Hence

$$\boxed{\begin{aligned} &\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + b_R = \rho \left.\frac{\partial v_R}{\partial t}\right|_{\underline{x}} \\ &\sigma_{RR} = t_R \qquad R = b \\ &\sigma_{RR} = -t_R \qquad R = a \end{aligned}}$$

5) Thermodynamics for Continua



Thermodynamics for a system (review)

- $\mathcal{E}$: Internal Energy
- S' : Entropy
- Ψ : Helmholtz free energy $E - \theta S'$
- θ : Temperature
- KE : Kinetic energy

State Variables

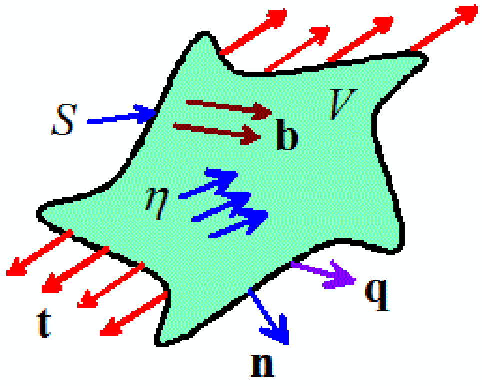
Rate of mechanical work W

Heat flux into system Q

First Law : $$\frac{d}{dt}\left(E + KE\right) = W + Q$$

Second Law $$\frac{dS'}{dt} \geq \frac{Q}{\theta}$$

Thermodynamics for Continua



Definitions:

ε - specific internal energy

s - " entropy

$\psi = \varepsilon - \theta s$

Volumetric heat flux η $\quad dQ = \eta \, dV$

Heat flux vector $\underline{q}$: $\quad dQ = \underline{q} \cdot \underline{n} \, dA$

gives heat flux across material surface element dA with normal $\underline{n}$

Themodynamic Laws

① First Law $\boxed{\rho \left.\frac{\partial \varepsilon}{\partial t}\right|_{\underline{x}} = \sigma : D + \eta - \nabla_y \cdot q}$

② Second Law $\rho \left.\frac{\partial s}{\partial t}\right|_{x} + \nabla_y \cdot \left(\frac{q}{\theta}\right) - \frac{\eta}{\theta} \geqslant 0$

(Clausius - Dubem inequality)

③ Dissipation Inequality :

$$\boxed{\sigma : D - \frac{1}{\theta} q \cdot \nabla_y \theta - \rho\left(\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t}\right) \geqslant 0}$$

Derivations

① First Law $\quad \frac{d}{dt}(\mathcal{E} + KE) = W + Q$

$$\frac{d}{dt}\left\{\int_V \rho\,\varepsilon\,dV + \int_V \tfrac{1}{2}\rho|\underline{v}|^2 dV\right\} = \int_V \rho\underline{b}\cdot\underline{v}\,dV + \int_S \underline{t}\cdot\underline{v}\,dA$$

$$\int_V \eta\,dV - \int_S \underline{q}\cdot\underline{n}\,dA$$

(a) Mechanical Work identity (See L15)

$$\int_V \rho\underline{b}\cdot\underline{v}\,dV + \int_S \underline{t}\cdot\underline{v}\,dA = \frac{d}{dt}\int_V \tfrac{1}{2}\rho|\underline{v}|^2 dV + \int_V \sigma : D\,dV$$

(b) Note $\quad \frac{d}{dt}\int_V \rho\,\varepsilon\,dV = \frac{d}{dt}\int_{V_0} \underbrace{\rho\,\varepsilon\,J}_{\rho J = \rho_0}\,dV_0 = \int_{V_0} \rho_0 \frac{\partial \varepsilon}{\partial t}\,dV_0$

$(\rho_0 = \text{const})$

$$= \int_{V_0} \frac{\partial \varepsilon}{\partial t} \rho J \, dV_0 = \int_V \rho \frac{\partial \varepsilon}{\partial t} \, dV$$

(c) Finally $\int_S \underset{\sim}{q} \cdot \underline{n} \, dA = \int_V \nabla_y \cdot \underset{\sim}{q} \, dV$

Collect all terms

$$\int_V \left(\rho \frac{\partial \varepsilon}{\partial t}\Big|_{\underline{x}} - \sigma : D + \nabla_y \cdot \underset{\sim}{q} - \eta \right) dV = 0$$

Must hold $\forall V \Rightarrow$ first Law.