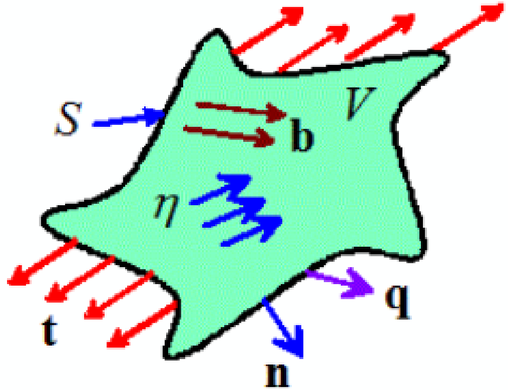
Review

Thermodynamic Laws for Continua



Define

- Specific internal energy ε
- Specific entropy s
- Specific Helmholtz free energy $\psi = \varepsilon - \theta s$
- External volumetric heat flux η
- Heat flux vector $\mathbf{q}$
- Cauchy Stress $\boldsymbol{\sigma}$
- Stretch Rate $\mathbf{D}$

First Law $\rho \dfrac{\partial \varepsilon}{\partial t} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\mathbf{y}} \cdot \mathbf{q} + \eta$

Second Law $\rho \dfrac{\partial s}{\partial t} + \nabla_{\mathbf{y}} \cdot \left(\dfrac{\mathbf{q}}{\theta} \right) - \dfrac{\eta}{\theta} \geq 0$

Dissipation Inequality $\boldsymbol{\sigma} : \mathbf{D} - \dfrac{1}{\theta} \mathbf{q} \cdot \nabla_{\mathbf{y}} \theta - \rho \left(\dfrac{\partial \psi}{\partial t} + s \dfrac{\partial \theta}{\partial t} \right) \geq 0$

Second Law $\quad \dfrac{dS}{dt} - \dfrac{Q}{\theta} \geq 0$

$$\Rightarrow \frac{d}{dt}\int_V \rho s \, dV - \int_V \frac{\eta}{\theta}\, dV + \int_S \frac{\underline{q}}{\theta}\cdot \underline{n}\, dA \geq 0$$

(a) Take time deriv into vol integral (same as for $\mathcal{E}$ in first law)

$$\frac{d}{dt}\int_V \rho s \, dV = \int_V \rho \frac{\partial s}{\partial t}\, dV$$

(b) Div. theorem $\Rightarrow \displaystyle\int_S \frac{\underline{q}\cdot\underline{n}}{\theta}\, dA = \int_V \nabla_y \cdot \left(\frac{\underline{q}}{\theta}\right) dV$

$$\Rightarrow \int_V \Big\{ \underbrace{\rho \frac{\partial s}{\partial t} - \frac{\eta}{\theta} + \nabla_y \cdot \left(\frac{\underline{q}}{\theta}\right)}_{\text{Local form } \geq 0} \Big\} dV \geq 0$$

Dissipation Inequality : start with second law

$$\rho \frac{\partial s}{\partial t} - \frac{\eta}{\theta} + \nabla_y \cdot \left(\frac{q}{\theta} \right) \geq 0$$

(a) $$\nabla_y \cdot \left(\frac{q}{\theta} \right) = \frac{1}{\theta} \nabla_y \cdot q - \frac{1}{\theta^2} q \cdot \nabla_y \theta$$

(b) $$\nabla_y \cdot q = \sigma : D + \eta - \rho \frac{\partial \varepsilon}{\partial t} \qquad \text{(first Law)}$$

$$\varepsilon = \psi + \theta s \qquad \Rightarrow \rho \frac{\partial \varepsilon}{\partial t} = \rho \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} + \theta \frac{\partial s}{\partial t} \right)$$

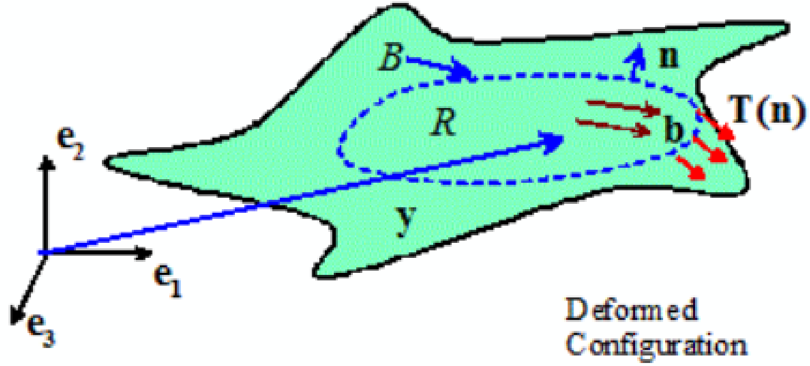
$$\Rightarrow \rho \cancel{\frac{\partial s}{\partial t}} - \cancel{\frac{\eta}{\theta}} + \frac{1}{\theta} \left\{ \sigma : D + \cancel{\eta} - \rho \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} + \cancel{\theta \frac{\partial s}{\partial t}} \right) - \frac{1}{\theta} q \cdot \nabla_y \theta \right\} \geq 0$$

since $\theta \geq 0$ we conclude ≥ 0

Conservation Laws for a control Volume

Define fixed region of space R with boundary B

R, B independent of time



Mass Conservation: $\frac{d}{dt}\int_R \rho dV + \int_B \rho \mathbf{v}\cdot\mathbf{n} dA = 0$

Linear Momentum Balance $\int_B \mathbf{n}\cdot\boldsymbol{\sigma} dA + \int_R \rho \mathbf{b} dV = \frac{d}{dt}\int_R \rho \mathbf{v} dV + \int_B (\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n} dA$

Angular Momentum Balance $\int_B \mathbf{y}\times(\mathbf{n}\cdot\boldsymbol{\sigma}) dA + \int_R \mathbf{y}\times(\rho\mathbf{b}) dA = \frac{d}{dt}\int_R \mathbf{y}\times\rho\mathbf{v} dV + \int_B (\mathbf{y}\times\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n} dA$

Mechanical Power Balance $\int_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v} dA + \int_R \rho\mathbf{b}\cdot\mathbf{v} dV = \int_R \boldsymbol{\sigma} : \mathbf{D} dV + \frac{d}{dt}\int_R \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v}) dV + \int_B \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})\mathbf{v}\cdot\mathbf{n} dA$

First law of thermodynamics

$$\int_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v} dA + \int_R \rho\mathbf{b}\cdot\mathbf{v} dV - \int_B \mathbf{q}\cdot\mathbf{n} dA + \int_V q dV = \frac{d}{dt}\int_R \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) dV + \int_B \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\cdot\mathbf{n} dA$$

Second law of thermodynamics

$$\frac{d}{dt}\int_R \rho s dV + \int_B \rho s(\mathbf{v}\cdot\mathbf{n}) dA + \int_B \frac{\mathbf{q}\cdot\mathbf{n}}{\theta} dA - \int_R \frac{q}{\theta} dV \geq 0$$

Proof of mass conservation: Start with local form

$$\left.\frac{\partial \rho}{\partial t}\right|_{y} + \nabla_y \cdot (\rho \underline{v}) = 0 \quad \text{(from earlier}$$

All functions of y: can integrate over fixed spatial volume:

$$\int_R \left.\frac{\partial \rho}{\partial t}\right|_y dV + \int_R \nabla_y \cdot (\rho \underline{v})\, dV = 0$$

R. fixed, apply div. thm

$$\frac{d}{dt}\int_R \rho\, dV + \int_B \rho\, \underline{v}\cdot\underline{n}\, dA = 0$$

Linear momentum: Local form

$$\nabla_y \cdot \sigma + \rho \underline{b} = \rho \frac{\partial \underline{v}}{\partial t}\Big|_{\underline{x}} = \rho \frac{\partial \underline{v}}{\partial t}\Big|_{y} + \rho (\nabla_y \underline{v}) \underline{v}$$

Now consider $\frac{\partial}{\partial t}(\rho \underline{v})\Big|_y + \nabla_y \cdot (\rho \underline{v} \otimes \underline{v}) =$

$$= \cancel{\frac{\partial \rho}{\partial t}\Big|_y v_i} + \rho \frac{\partial v_i}{\partial t}\Big|_y + \underbrace{\frac{\partial}{\partial y_j}(\rho v_i v_j)}$$

$$= \rho \frac{\partial v_i}{\partial y_j} v_j + \cancel{\frac{\partial \rho v_j}{\partial y_j}} v_i$$

cancel from mass conservation

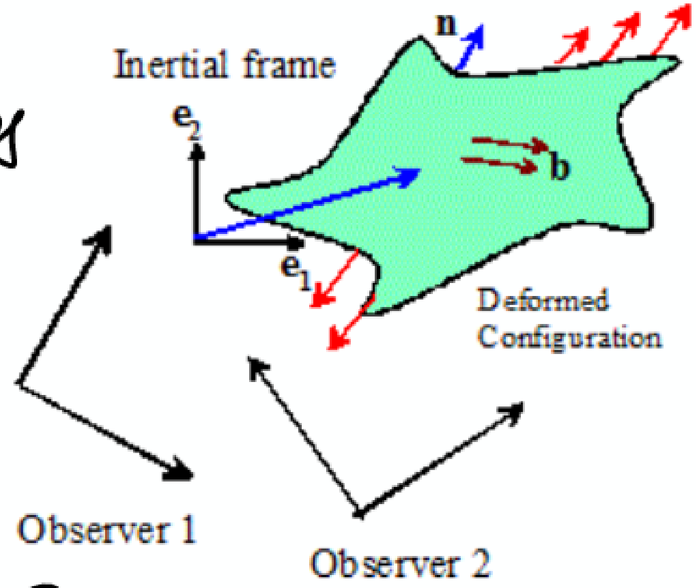
Collect terms and integrate over R

$$\int_R \nabla_y \cdot \sigma \, dV + \int_R \rho \underline{b} \, dV = \int_R \frac{\partial}{\partial t} \rho \underline{v} \, dV + \int_R \nabla_y \cdot (\rho \underline{v} \otimes \underline{v}) dV$$

Apply divergence theorem to first & last terms
take time deriv outside integral QED

6 Frame Indifference or "objectivity" in continuum mechanics

Introduce "observers": physical quantities in Inertial frame are measured by two different observers



At time $t=0$ both observers have same position & orientation

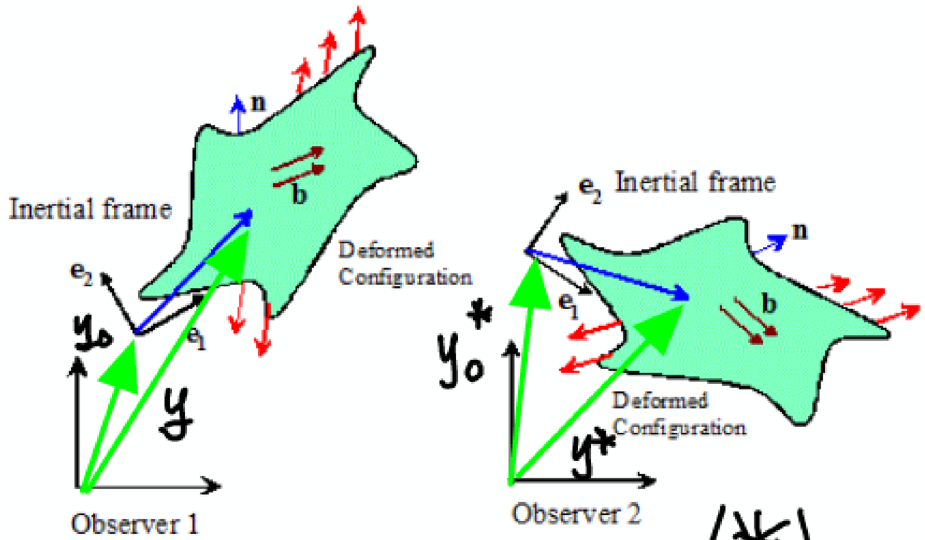
Both observers use continuum @ $t=0$ as reference, and assume ref config is fixed

Observers are in relative motion

Observer Transformations

Consider some physical vector
eg relative position of two
points

Transformation from one
observer to the other is
a length and angle preserving
map



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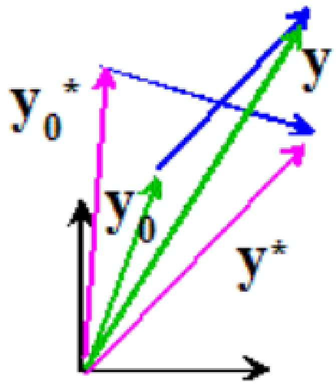

$$(y^* - y_0^*) = Q(t)\,(y - y_0)$$

$Q(t)$ is othogonal

$$QQ^T = Q^TQ = I$$