Review: Frame changes

Observers:

Physical vectors in an inertial frame can be measured by observers in relative motion.

Observer transformations:

A vector connecting two points in the inertial frame (e.g., a vector from one point in the frame to another) transforms as

\[ y^* - y_0^* = Q(t)(y - y_0) \]

\[ QQ^T = Q^T Q = I \]
Objective Vectors

Some physical quantities are properties of the inertial frame (independent of choice of ref config).

- Examples: relative position, body force, traction vector normal to a surface

These vectors can be regarded as connecting two points in space.

They transform as \( \mathbf{u}^* = Q \mathbf{u} \) (1)

A vector satisfying (1) is called "objective."
Not all vectors are objective: eg velocity

\[ \dot{v} = \frac{d}{dt} (y - y_0) \quad \dot{v}^* = \frac{d}{dt} (y^* - y_{0}^*) \]

\[ = \frac{d}{dt} Q (y - y_0) \]

\[ = Q \frac{d}{dt} (y - y_0) + \frac{d}{dt} Q (y - y_0) \]

\[ = Q \dot{v} + \frac{d}{dt} \dot{Q} Q^T (y^* - y_{0}^*) \]

\( \dot{\omega} \) - frame spin: \( \dot{\omega} = -\omega^T \)

\[ \frac{d}{dt} (Q Q^T) = \frac{d}{dt} (I) = 0 \quad Q Q^T + Q^T \dot{Q} = 0 \]

\[ \dot{L} + \dot{\omega}^T = 0 \]
**Objective Tensors**: map an objective vector onto another objective vector

For an objective tensor \( S^* = Q S Q^T \)

To see this, let \( \mathbf{u}, \mathbf{v} \) be two objective vectors with \( \mathbf{v} = S \mathbf{u} \)

\[
\mathbf{v}^* = S^* \mathbf{u}^* = Q \mathbf{v} = S^* Q \mathbf{u}
\]

\[
\Rightarrow \mathbf{v} = Q^T S^* Q \mathbf{u}
\]

\[
S = Q^T S^* Q \Rightarrow Q S Q^T = S^*
\]

**Invariant Tensors**:

A tensor is invariant if \( S^* = S \)
Transformation rules for common tensors in mechanics

Deformation gradient \( F = \nabla (y - y_0) \)

\[ F^* = \nabla (y^* - y_0^*) = Q \nabla (y - y_0) = QF \]

\[ F^* = QF \]

Cauchy-Green tensors

\[ C = F^T F \quad C^* = F^{*T} F^* = F^T Q^T Q F = F^T F = C \]

\[ C = C^* \text{ invariant} \]

This is because \( C \) quantifies length changes of fiber in ref config
\[ \frac{\varepsilon^2}{\varepsilon_0^2} = n \cdot \frac{C_m}{m} \quad \text{where } m \text{ is a reference fiber} \]

\[ B = FF^T \quad B^* = F^* F^{*T} = QF F^T Q^T = QBQ^T \]

**B** is a property of current config:

\[ \frac{\varepsilon_0^T}{E^T} = \Omega \cdot B^{-1} \Omega \quad \text{in material fiber} \]

in current config

**Measures of deformation rate:**

\[ \dot{\varepsilon} = \nabla_y Y = F^T \dot{F} \Rightarrow \dot{\varepsilon}^* = F^* F^{*T} \dot{F} \]

\[ \dot{\varepsilon}^* = (QF + QF^T) F^{-1} Q^T \]

\[ = \Omega + Q \varepsilon Q^T \]
Stretch rate tensor \( D = \text{sym} (\dot{\varepsilon}) = \frac{1}{2} (\dot{\varepsilon} + \dot{\varepsilon}^T) \)

\[ D^* = QDQ^T \quad \text{since } Q \text{ is skew} \]

Spin tensor \( W = \text{skew} (\dot{\varepsilon}) = \frac{1}{2} (\dot{\varepsilon} - \dot{\varepsilon}^T) \)

\[ W^* = QWQ^T + I \]

Polar Decomposition \( F = RU = VR \)

\[ U = C^{1/2} \rightarrow U = U^* \]

Hence \( F^* = R^* U^* \rightarrow QF = R^* U \)

\[ QFU^{-1} = R^* \]

\[ QR = R^* \]

Also \( V = RUR^T \rightarrow V^* = R^* U^* R^T \)
\[ V^* = QRU R^T Q^T = Q V Q^T \]

**Stress Measures:**

**Cauchy Stress:** \( \mathbf{T} (n) = n \sigma \)

\( n \), in objective vectors \( \Rightarrow \sigma \) is objective

\[ \sigma^* = Q \sigma Q^T \]

**Nominal Stress** \( S = J F^{-1} \sigma \)

**Note** \( J^* = \det (F^*) = \det (QF) = \det (F) = J \)

\[ S^* = J^* F^{-1} \sigma^* = J F^{-1} Q^T Q \sigma Q^T \]

\[ = S Q^T \]
Material Stress $\varepsilon = J F^{-1} \sigma F^{-T}$

$\varepsilon^* = J^* F^{*-1} \sigma^* F^{*-T}$

$= J F^{-1} Q^T Q \sigma Q^T Q F^{-T} = \varepsilon$