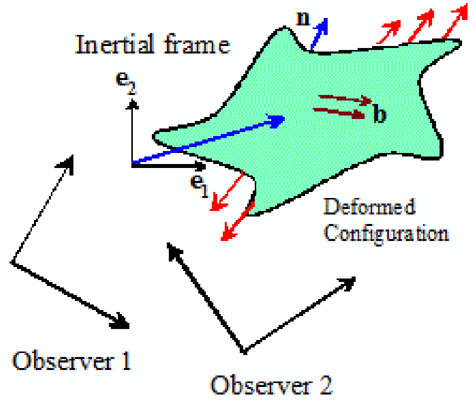
Review: Frame changes



Observers:

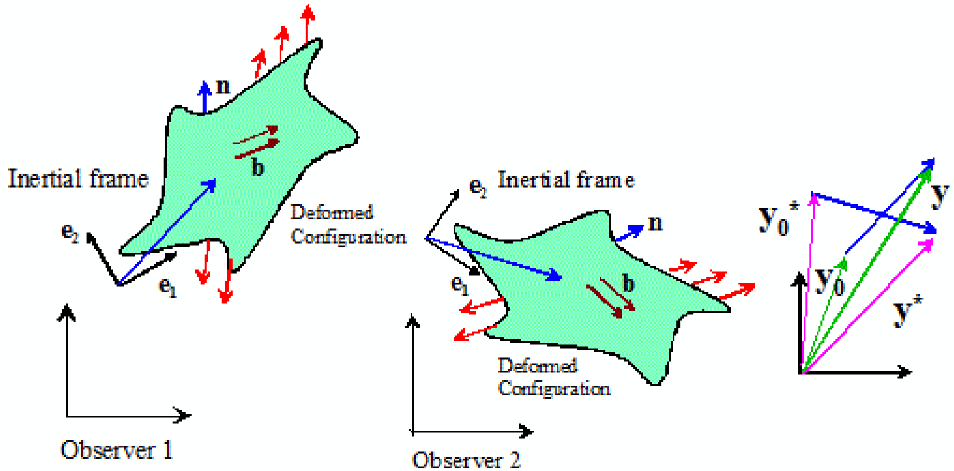
Physical vectors in an inertial frame can be measured by observers in relative motion

Observer transformations:

A vector connecting two points in the inertial frame (eg a vector from one point in the frame to another) transforms as

$$\mathbf{y}^* - \mathbf{y}_0^* = \mathbf{Q}(t)\left(\mathbf{y} - \mathbf{y}_0\right)$$

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$$



Objective Vectors

Some physical quantities are properties of the inertial frame (independent of choice of ref config)

- Examples: relative position vector, body force, traction normal to a surface

These vectors can be regarded as connecting two points in space

They transform as $\underline{u}^* = Q\underline{u}$ (1)

A vector satisfying (1) is called "objective"

Not all vectors are objective : eg velocity

$$\underline{v} = \frac{d}{dt}(\underline{y} - \underline{y}_0) \qquad \underline{v}^* = \frac{d}{dt}(\underline{y}^* - \underline{y}_0^*)$$

$$= \frac{d}{dt} Q(\underline{y} - \underline{y}_0)$$

$$= Q\frac{d(\underline{y} - \underline{y}_0)}{dt} + \frac{d}{dt}Q(\underline{y} - \underline{y}_0)$$

$$= Q\underline{v} + \underbrace{\frac{dQ}{dt} Q^T}_{\Omega}(\underline{y}^* - \underline{y}_0^*)$$

Ω - frame spin : $\Omega = -\Omega^T$

$$\frac{d}{dt}(Q Q^T) = \frac{d}{dt}(I) = 0 \qquad \dot{Q} Q^T + Q \dot{Q}^T = 0$$

$$\Omega + \Omega^T = 0$$

Objective Tensors : map an objective vector onto another objective vector

For an objective tensor $S^* = Q S Q^T$

To see this let $\underline{u}$, $\underline{v}$ be two objective vectors with $\underline{v} = S\underline{u}$

$$\underline{v}^* = S^* \underline{u}^* = Q\underline{v} = S^* Q \underline{u}$$

$$\Rightarrow \underline{v} = \underbrace{Q^T S^* Q}\ \underline{u}$$

$$S = Q^T S^* Q \Rightarrow Q S Q^T = S^*$$

Invariant tensors :

A tensor is invariant if $S^* = S$

Transformation rules for common tensors in mechanics

Deformation gradient

$$F = \nabla(y - y_0)$$

$$F^* = \nabla(y^* - y_0^*) = Q\nabla(y - y_0) = QF$$

$$F^* = QF$$

Cauchy-Green tensors

$$C = F^T F \qquad C^* = F^{*T} F^* = F^T Q^T Q F = F^T F = C$$

$$C = C^* \quad \text{invariant}$$

This is because C quantifies length changes of fiber in ref config

$$\frac{\ell^2}{\ell_0^2} = \underset{\sim}{m} \cdot C \underline{m}$$

$\ell_0 \underline{m}$ is a reference fiber

$$B = FF^T \qquad B^* = F^* F^{*T} = QFF^TQ^T = QBQ^T$$

B is a property of current config:

$$\frac{\ell_0^2}{\ell^2} = \underline{n} \cdot B^{-1} \underline{n}$$

$\ell \underline{n}$ material fiber in current config

Measures of deformation rate:

$$L = \nabla_y \underline{V} = \dot{F} F^{-1} \Rightarrow L^* = \dot{F}^* F^{*-1}$$

$$L^* = (\dot{Q} F + Q \dot{F}) F^{-1} Q^T$$

$$= \Omega + Q L Q^T$$

Stretch rate tensor $D = \text{sym}(L) = \frac{1}{2}(L + L^T)$

$$D^* = QDQ^T \quad \text{since } \Omega \text{ is skew}$$

Spin tensor $W = \text{skew}(L) = \frac{1}{2}(L - L^T)$

$$W^* = QWQ^T + \Omega$$

Polar Decomposion $F = RU = VR$

$$U = C^{1/2} \Rightarrow U = U^*$$

Hence $F^* = R^* U^* \Rightarrow QF = R^* U$

$$QFU^{-1} = R^*$$

$$QR = R^*$$

also $V = RUR^T \Rightarrow V^* = R^* U^* R^{*T}$

$$V^* = Q R U R^T Q^T = Q V Q^T$$

Stress Measures :

Cauchy Stress : $\underline{T}(\underline{n}) = \underline{n}\,\sigma$

$\underline{T}$, $\underline{n}$ objective vectors $\Rightarrow$ σ is objective

$$\sigma^* = Q \sigma Q^T$$

Nominal Stress $S = J F^{-1} \sigma$

Note $J^* = \det(F^*) = \det(QF) = \det(F) = J$

$$S^* = J^* F^{*-1} \sigma^* = J F^{-1} Q^T Q \sigma Q^T$$
$$= S Q^T$$

Material Stress $\quad \Sigma = J F^{-1} \sigma F^{-T}$

$$\Sigma^* = J^* F^{*-1} \sigma^* F^{*-T}$$

$$= J F^{-1} Q^T Q \sigma Q^T Q F^{-T} = \Sigma$$