Review – vectors and index notation

Point: $\mathbf{x} \in \mathbb{R}^3 \quad (x_1, x_2, x_3) \equiv x_i \qquad i = 1, 2, 3$

Metric: $g(\mathbf{x}, \mathbf{y}) = \sqrt{(x_i - y_i)(x_i - y_i)} \equiv \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$

Vector: $\mathbf{v} = \mathbf{x} - \mathbf{y} \quad \forall \quad [\mathbf{x}, \mathbf{y} \in \mathbb{R}^3] \qquad v_i = x_i - y_i$

Dot Product: $\mathbf{v} \cdot \mathbf{w} \equiv v_i w_i \equiv v_1 w_1 + v_2 w_2 + v_3 w_3$

Cross Product: $[\mathbf{v} \times \mathbf{w}]_i \equiv \epsilon_{ijk} v_j w_k$

$$\epsilon_{ijk} \equiv \begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \end{cases}$$

Linearly Independent Vectors: $\mathbf{m}_i : \qquad c_i \mathbf{m}_i \neq 0 \quad \forall \quad c_i \neq 0$

Basis: any 3 linearly independent vectors

Vector Components: $\mathbf{v} = v_i^{(\mathbf{m})} \mathbf{m}_i \equiv v_1^{(\mathbf{m})} \mathbf{m}_1 + v_2^{(\mathbf{m})} \mathbf{m}_2 + v_3^{(\mathbf{m})} \mathbf{m}_3$

$$\mathbf{m}_k \cdot \mathbf{v} = v_i^{(\mathbf{m})} \mathbf{m}_k \cdot \mathbf{m}_i = g_{ki} v_i^{(\mathbf{m})} \qquad g_{ki} = \mathbf{m}_k \cdot \mathbf{m}_i$$

$$v_i^{(\mathbf{m})} = g_{ik}^{-1} \mathbf{m}_k \cdot \mathbf{v}$$

[Index Notation A_{ij} - 3x3 matrix

* Matrix - Vector products

$$c_i = A_{ij}\, b_j$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & & \\ A_{31} & & \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

* Matrix products

$$[C] = [A][B] \equiv C_{ij} = A_{ik} B_{kj}$$

$$[C] = [A]^T[B] \equiv C_{ij} = A_{ki} B_{kj}$$

$$[C] = [A][B]^T \equiv C_{ij} = A_{ik} B_{jk}$$

* Inverse

$$[A]^{-1}[A] = [I]$$

$$A^{-1}_{ij} A_{jk} = \delta_{ik}$$

* δ_{ij} – Kronecker Delta $\qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & j \neq i \end{cases}$

Example using index notation

$$g_{ki} V_i^{(\underline{m})} = \underline{m}_k \cdot \underline{V}$$

$$\overset{-1}{g}_{jk} g_{ki} V_i^{(\underline{m})} = \overset{-1}{\underline{g}}_{jk} \underline{m}_k \cdot \underline{V}$$

$$\delta_{ji} V_i^{(\underline{m})} = \overset{-1}{g}_{jk} \underline{m}_k \cdot \underline{V}$$

$$V_j^{(\underline{m})} = \overset{-1}{g}_{jk} \underline{m}_k \cdot \underline{V}$$

Special Bases :

(1) Standard basis $\underset{\sim}{e}_1 = (1\ 0\ 0)$

$\underset{\sim}{e}_2 = (0\ 1\ 0)$

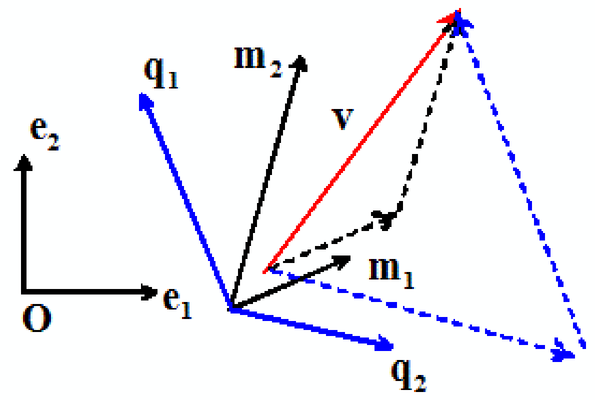
$\underset{\sim}{e}_3 = (0\ 0\ 1)$

(2) Othonormal basis $\underset{\sim}{m}_i \cdot \underset{\sim}{m}_j = \delta_{ij}$

Basis Change Formulas

Let $\left.\begin{matrix} \underset{\sim}{q}_i \\ \underset{\sim}{m}_i \end{matrix}\right\}$ be two bases

$\left.\begin{matrix} V_i^{\underset{\sim}{q}} \\ V_i^{\underset{\sim}{m}} \end{matrix}\right\}$ be components



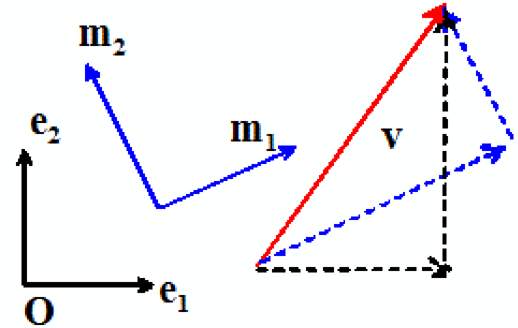
Relate the two

$$\underline{V} = V_i^{\underline{m}}\, \underline{m}_i = V_i^{q}\, \underline{g}_i \quad *$$

$$\underline{m}_k \cdot (*) \Rightarrow g_{ki} V_i^{\underline{m}} = h_{ki} V_i^{q} \qquad g_{ki} = \underline{m}_k \cdot \underline{m}_i$$

$$V_j^{\underline{m}} = g_{jk}^{-1} h_{ki} V_i^{(q)} \qquad h_{ki} = \underline{m}_k \cdot \underline{g}_i$$

Special Case: Orthonormal Bases



$$\underline{V} = V_i^{\underline{m}}\, \underline{m}_i = V_i^{\underline{e}}\, \underline{e}_i \quad *$$

$$\underline{m}_k \cdot (*) \Rightarrow \underline{m}_k \cdot \underline{m}_i\, V_i^{\underline{m}} = \underline{m}_k \cdot \underline{e}_i\, V_i^{\underline{e}}$$

$$\underline{e}_k \cdot (*) \Rightarrow V_k^{\underline{m}} = Q_{ki} V_i^{\underline{e}} \qquad Q_{ki} = \underline{m}_k \cdot \underline{e}_i$$

$$Q_{ik} V_i^{\underline{m}} = V_k^{\underline{e}}$$

Properties of Q_{ij} : * $\underline{m}_k = Q_{kj}\,\underline{e}_j$

* $Q_{ki}\,Q_{kj} = Q_{ik}\,Q_{jk} = \delta_{ij}$

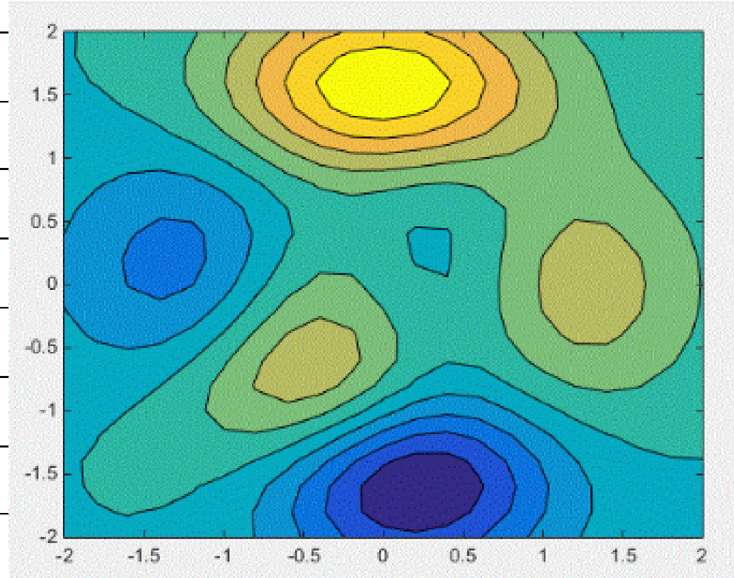
$$[Q]^T[Q] = [Q][Q]^T = [I]$$

To see this note:

$$V_k^{\underline{m}} = Q_{ki}\,Q_{ji}\,V_j^{\underline{m}}$$

$$\underbrace{[\delta_{kj} - Q_{ki}\,Q_{ji}]}_{=0}\,V_j^{\underline{m}} = 0 \quad \forall\, V_j^{\underline{m}}$$

Vector Calculus



* Scalar Field $\phi(x_1, x_2, x_3)$

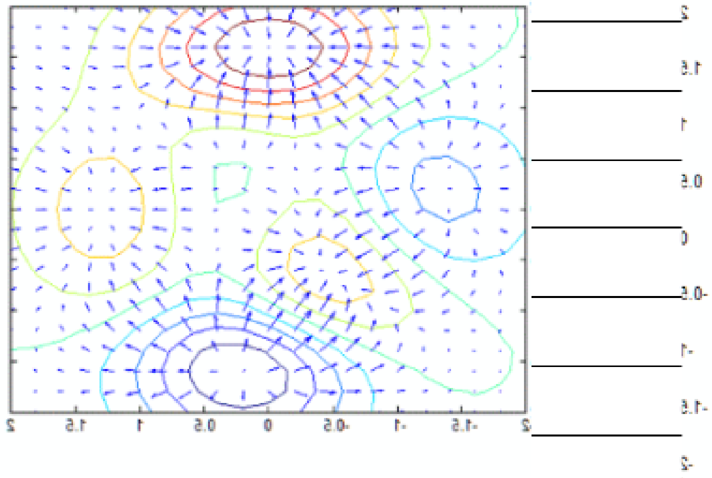
* Gradient

$$\nabla \phi = \frac{\partial \phi}{\partial x_i}$$

Unit Vector

Or $\nabla \phi \cdot \underline{n} = \lim_{\epsilon \to 0} \frac{\phi(\underline{x} + \epsilon \underline{n}) - \phi(\underline{x})}{\epsilon}$

$$\phi(\underline{x} + d\underline{x}) = \phi(\underline{x}) + \nabla \phi \cdot d\underline{x} + O(|d\underline{x}|)^2$$

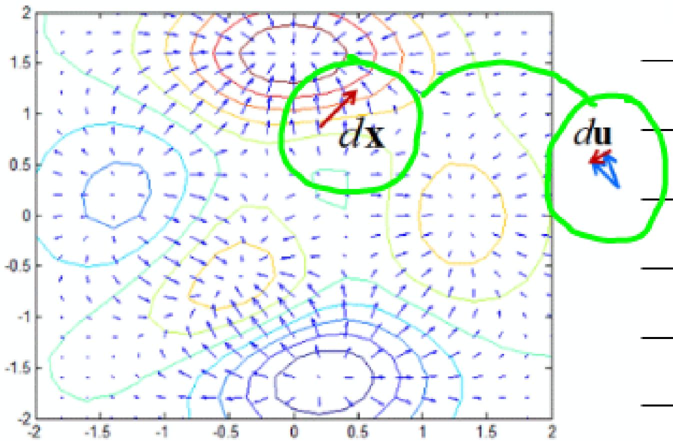


* Vector field $\underline{u}\,(x_1, x_2, x_3)$

eg $\underline{u} = \nabla \phi$

* Gradient $[\nabla \underline{u}]_{ij} = \dfrac{\partial u_i}{\partial x_j}$

3x3 matrix $\begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \ddots & \\ \dfrac{\partial u_3}{\partial x_1} & & \text{etc} \end{bmatrix}$



Some authors use a $(\cdot)$

Or $\nabla \underline{u} \cdot \underline{n} = \lim_{\epsilon \to 0} \dfrac{\underline{u}(\underline{x} + \epsilon \underline{n}) - \underline{u}(\underline{x})}{\epsilon}$

Unit Vector

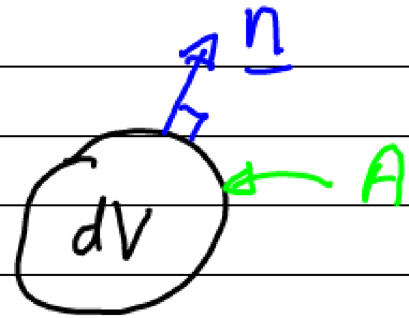
$$\underline{u}(\underline{x}+d\underline{x}) = \underline{u}(\underline{x}) + \nabla\underline{u}\, d\underline{x} + O(|d\underline{x}|^2)$$

* Divergence

$$\nabla\cdot\underline{u} = \text{trace}(\nabla\underline{u})$$

$$\nabla\cdot\underline{u} \equiv \frac{\partial u_i}{\partial x_i}$$

$$\nabla\cdot\underline{u} = \lim_{dV\to 0}\int_{\underline{A}} \underline{u}\cdot\underline{n}\, dA$$



* Curl $\quad [\nabla\times\underline{u}]_i \equiv \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}$

Note: some authors use $\underline{u}\overleftarrow{\nabla} \equiv \nabla \underline{u}$

Also $\vec{\nabla} \underline{u} \equiv [\nabla u]^T$ in some texts