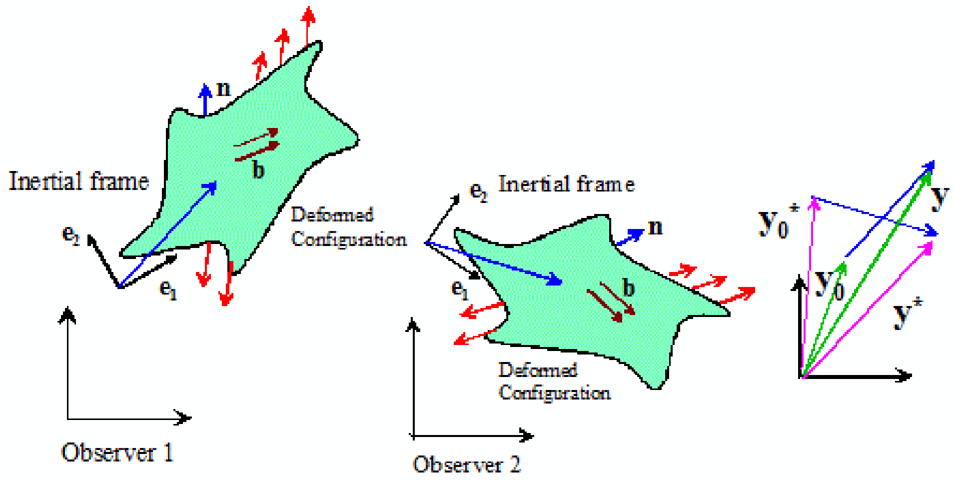
Review: Transformation of variables under frame changes

Observer transformations:

A vector connecting two points in the inertial frame (eg a vector from one point in the frame to another) transforms as

$$\mathbf{y}^* - \mathbf{y}_0^* = \mathbf{Q}(t)\left(\mathbf{y} - \mathbf{y}_0\right)$$

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$$



Transformation relations for common variables

Deformation measures:

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}$$

$$\mathbf{C}^* = \mathbf{F}^{*T}\mathbf{F}^* = \mathbf{C}$$

$$\mathbf{B}^* = \mathbf{F}^*\mathbf{F}^{*T} = \mathbf{Q}\mathbf{B}\mathbf{Q}^T$$

$$\mathbf{L} = \dot{\mathbf{F}}^*\mathbf{F}^{*-1} = \mathbf{Q}\mathbf{L}\mathbf{Q}^T + \mathbf{\Omega}$$

$$\mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^T \qquad \mathbf{W}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \mathbf{\Omega}$$

Stress measures:

$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T$$

$$\mathbf{S}^* = \mathbf{S}\mathbf{Q}^T$$

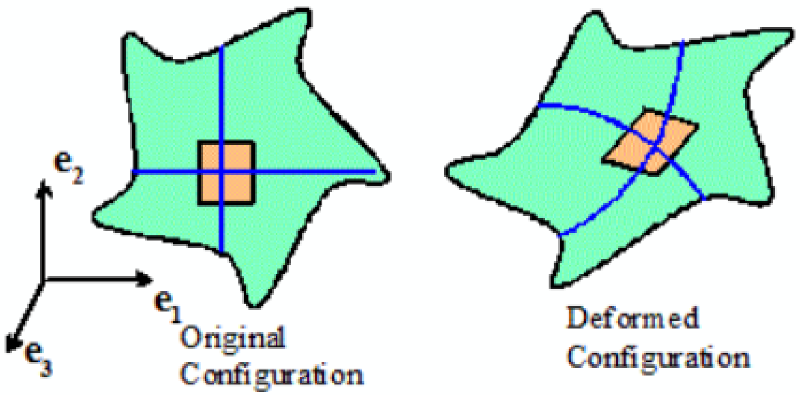
$$\mathbf{\Sigma}^* = \mathbf{\Sigma}$$

5) Constitutive Equations.

Goal: Relate deformation measures to stresses ($\sigma = E\varepsilon$)

Assumptions :

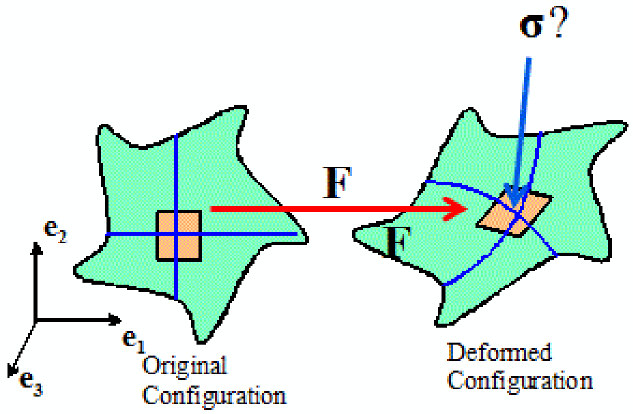
(a) Material is locally homogeneous
material in infinitesimal vol element
is uniform

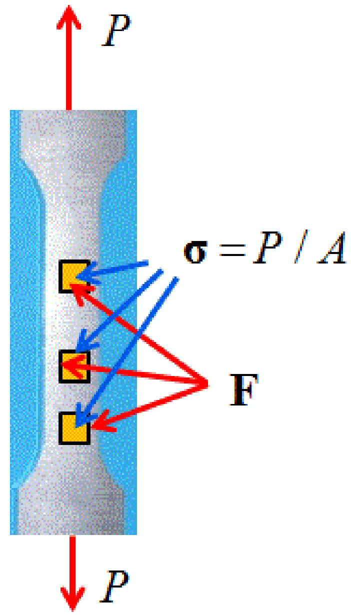


(b) Deformation is locally homogeneous
Continuous and differentiable lines in ref config
remain continuous and differentiable

(c) "Principle of local action" stress and internal variables at a point are determined by deformation of the immediate neighborhood

Consequences: stresses must depend on F or strain measures derived from it.





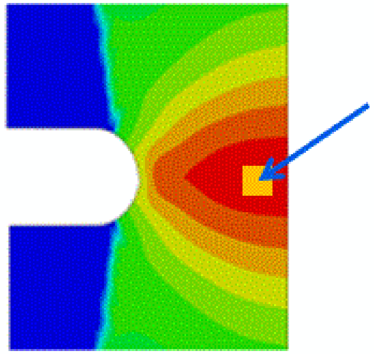
In a specimen with uniform F the stress will be uniform

Uniform stress can be calculated from forces applied to specimen

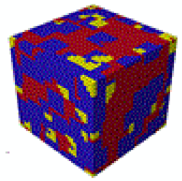
Constitutive law is a curve fit to experimental data

(Exception: elastic properties single xtal can be calculated from ab-initio simulations)

What does "local homogeneity" mean for real materials?



$\bar{\mathbf{F}}, \bar{\boldsymbol{\sigma}}$



$$\bar{\mathbf{F}} = \frac{1}{V}\int_V \mathbf{F}\,dV$$

$$\bar{\boldsymbol{\sigma}} = \frac{1}{V}\int_V \boldsymbol{\sigma}\,dV$$

All materials have some structure @ short scales (atoms, or microstructure)

At macroscopic length scales a "point" is made up of a large number of statistically identical "representative volume elements"

Constitutive law relates volume averaged deformation to vol averaged stress in RVE

Structure of constitutive equation

* Independent Variables: Deformation Measures (DM)
  - F, B, C, D, E, E^*

  Temperature θ

* Dependent variables: Thermodynamic functions ε, s, ψ

  Stress - σ, S, Σ

  Heat flux q

  State Variables: eg accumulated plastic strain; porosity, damage variables (SV)

Constitutive equations will specify (eg)

(1) Thermodynamic state vars, usually Helmholtz free energy $\psi = \hat{\psi}(DM, \theta, SV)$

(2) Stress response function $\sigma = \hat{\sigma}(DM, \theta, SV)$

$\Sigma = \hat{\Sigma}(DM, \theta, SV)$

(3) Heat flux response function

$q = \hat{q}(\theta, DM, SV)$

(4) Evolution equations for state variables

General Constraints on constitutive equations

(a) Thermodynamic Laws
(b) Material Frame Indifference

(c) "Drucker Stability" condition (optional, but materials violating this behave badly in calculations)

Thermodynamics:

1st Law $\rho \frac{\partial \varepsilon}{\partial t} = \eta + \sigma : D - \nabla_y \cdot q$ $\forall$ possible processes

We regard η to be arbitrarily assignable
- does not restrict behavior

2^{nd} Law: $$\sigma : D - \frac{1}{\theta} q \cdot \nabla_y \theta - \rho \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0$$

for all processes

- for purely mechanical theories

$$\sigma : D - \rho \frac{\partial \psi}{\partial t} \geq 0$$