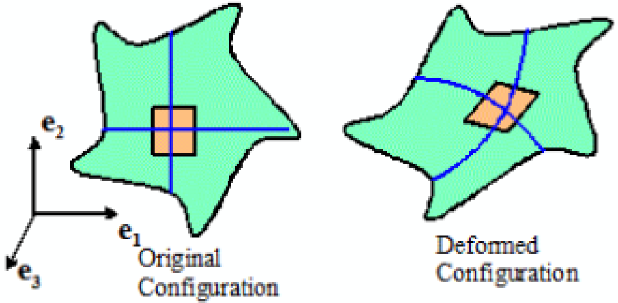
Review: Constitutive Equations

Assumptions:

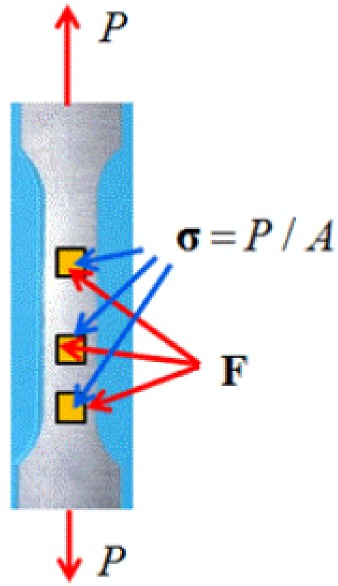


- Local Homogeneity of Material
- Local homogeneity of deformation
- Principle of Local Action

Implications:

- Stress measures depend on **F** or something calculated from **F**
- Can measure material response by subjecting it to uniform **F** and measuring forces

Structure of constitutive equations:



Independent Variables:

- Deformation measures **(F, B, C, E, D)** etc
- Temperature θ

Equations specify

- Thermodynamic state variables $\psi = \hat{\psi}(DM, \theta, SV)$
- Stress response function $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(DM, \theta, SV)$
- Heat transfer response function $\mathbf{q} = \hat{\mathbf{q}}(DM, \theta, SV)$
- Evolution equations for material state variables

Restrictions

- Thermodynamic Laws
- Frame Indifference
- Drucker Stability

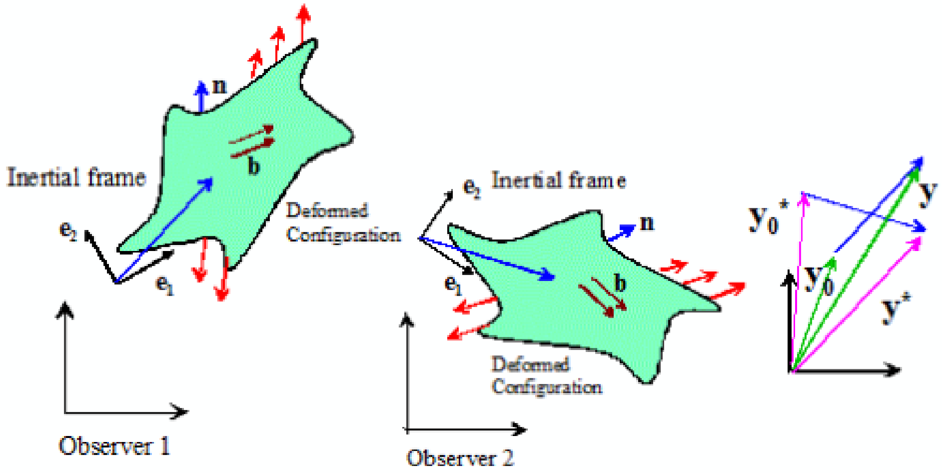
Review: Transformation of variables under frame changes

Observer transformations:

A vector connecting two points in the inertial frame (eg a vector from one point in the frame to another) transforms as

$$\mathbf{y}^* - \mathbf{y}_0^* = \mathbf{Q}(t)(\mathbf{y} - \mathbf{y}_0)$$

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$$



Transformation relations for common variables

Deformation measures:

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}$$

$$\mathbf{C}^* = \mathbf{F}^{*T}\mathbf{F}^* = \mathbf{C}$$

$$\mathbf{B}^* = \mathbf{F}^*\mathbf{F}^{*T} = \mathbf{Q}\mathbf{B}\mathbf{Q}^T$$

$$\mathbf{L}^* = \dot{\mathbf{F}}^*\mathbf{F}^{*-1} = \mathbf{Q}\mathbf{L}\mathbf{Q}^T + \mathbf{\Omega}$$

$$\mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^T \quad \mathbf{W}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \mathbf{\Omega}$$

Stress measures:

$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T$$

$$\mathbf{S}^* = \mathbf{S}\mathbf{Q}^T$$

$$\mathbf{\Sigma}^* = \mathbf{\Sigma}$$

$$\Omega = \dot{Q}Q^T \qquad \Omega = -\Omega^T$$

Frame Indifference

Assumption: Observers in relative motion will observe identical physical behavior

Constitutive equations must transform consistently under observer changes

Example: suppose

$$\psi = \hat{\psi}(F)$$
$$\psi^* = \hat{\psi}(F^*)$$

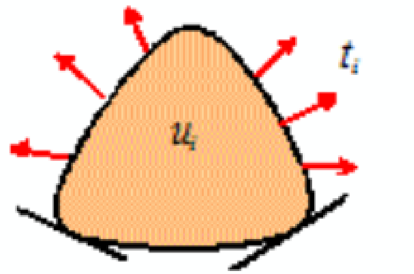
We assume $\psi = \psi^* \Rightarrow \hat{\psi}(F) = \hat{\psi}(QF)$

Alternatively

$$\sigma = \hat{\sigma}(F)$$
$$\sigma^* = \hat{\sigma}(F^*)$$

$$\sigma^* = Q\sigma Q^T \Rightarrow \hat{\sigma}(QF) = Q\,\hat{\sigma}(F)\,Q^T$$

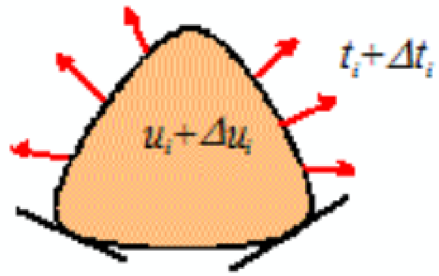
Drucker Stability



Suppose a material is subjected to a prescribed (equilibrium) traction $\underset{\sim}{t}$

They produce a displacement $\underset{\sim}{u}$

Suppose $\underset{\sim}{t} \to \underset{\sim}{t} + \Delta \underset{\sim}{t}$ quasi-static
producing $\underset{\sim}{u} \to \underset{\sim}{u} + \Delta \underset{\sim}{u}$



$$\Delta W = \int_S \Delta \underset{\sim}{t} \cdot \Delta \underset{\sim}{u} \, dA$$

$\Delta W \geq 0$ for all $\Delta \underset{\sim}{t}, \Delta \underset{\sim}{u} \Rightarrow$ "Drucker stable"

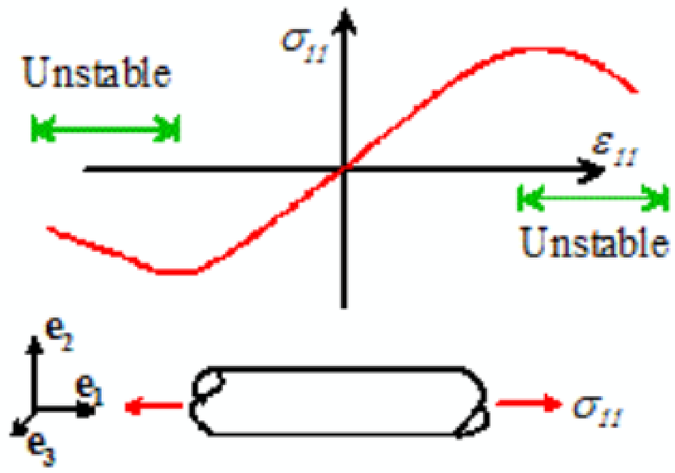
This requires $\Delta\sigma_{ij}\frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial y_j} + \frac{\partial \Delta u_j}{\partial y_i}\right) \geq 0$

or $\Delta\sigma_{ij}\,\Delta\varepsilon_{ij}' \geq 0$

* Example of a material that is not drucker stable:

strain softening

Here $\Delta\sigma_{11} < 0$ when $\Delta\varepsilon_{11} > 0$



These materials "behave badly"
- nonunique solutions to BVPs
- often localize - velocity discontinuities (violates an assumption)

Example: Constitutive Laws for fluids

A fluid: (1) No natural reference config (except density)
(2) Fluid that is stationary supports no shear stress

$$\sigma = pI$$

Goal: Construct constitutive laws

(1) $\Rightarrow$ constitutive functions can only depend on deformation measures that are independent of ref config eg $L = \nabla_y v$, D, W

Hence we suppose that thermodynamic functions have form

eg $\hat{\psi}(\rho, \theta, L)$ (neglect state vars)

or $\hat{\sigma}(\rho, \theta, L)$ etc

We will show that:

(1) Thermodynamic functions have form

$\hat{\varepsilon}(\rho, \theta)$ - internal energy

$\hat{s}(\rho, \theta)$ - entropy

$\hat{\psi}(\rho, \theta)$ - Helmholtz free energy

Also define

$$c_v = \frac{\partial \hat{\varepsilon}}{\partial \theta}$$

(2) Stress response function:

$$\hat{\sigma} = -\Pi_{eq}(\rho, \theta) I + \hat{\sigma}^{vis}(\rho, \theta, D)$$

where $\hat{\sigma}^{vis}(\rho, \theta, D=0) = 0$

Also $\hat{\sigma}^{vis}$ must satisfy $Q \hat{\sigma}^{vis}(D) Q^T = \hat{\sigma}^{vis}(Q D Q^T)$

This requires $\hat{\sigma}^{vis} = f_1 I + f_2 D + f_3 D^2$

f_1, f_2, f_3 are functions of eigenvalues (or invariants) of D

Also show that:

$$\hat{\pi}_{eq} = \rho^2 \frac{\partial \hat{\psi}}{\partial \rho} \qquad \hat{s} = -\frac{\partial \psi}{\partial \theta}$$

$$\frac{\partial \hat{\pi}_{eq}}{\partial \theta} = -\rho^2 \frac{\partial \hat{s}}{\partial \rho} \qquad \hat{\pi}_{eq} = \theta \frac{\partial \hat{\pi}_{eq}}{\partial \theta} + \rho^2 \frac{\partial \hat{\varepsilon}}{\partial \rho}$$

$$c_v = -\theta \frac{\partial^2 \hat{\psi}}{\partial \theta^2} \qquad \frac{\partial c_v}{\partial \rho} = -\frac{\theta}{\rho^2} \frac{\partial^2 \hat{\pi}_{eq}}{\partial \theta^2}$$

Start with Helmholtz free energy

Suppose $\psi = \hat{\psi}(\rho, \theta, L)$

Frame Indifference $\Rightarrow$ $\hat{\psi}(\rho, \theta, L) = \hat{\psi}(\rho, \theta, QLQ^T + \Omega)$

Must hold for all Q, for all Ω

Recall $L = D + W$

$$\hat{\psi}(\rho, \theta, D + W) = \hat{\psi}(\rho, \theta, QDQ^T + QWQ^T + \Omega)$$

Choose $Q = I$, $\Omega = -W$

Hence $\hat{\psi}(\rho, \theta, L) = \hat{\psi}(\rho, \theta, D)$

Similar argument shows that:

$$\sigma = \hat{\sigma}(\rho, \theta, D)$$

$$q = \hat{q}(\rho, \theta, D)$$