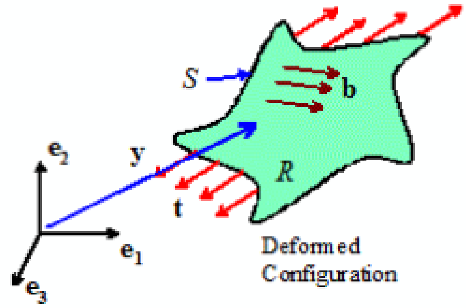
Review – Constitutive Models for Fluids



Properties of fluids

- No natural reference configuration
- Support no shear stress when at rest

Goal: show that constitutive models for fluids must have the following structure

Internal Energy $\varepsilon = \hat{\varepsilon}(\rho, \theta)$

Entropy $s = \hat{s}(\rho, \theta)$

Free Energy $\psi = \hat{\psi}(\rho, \theta) = \varepsilon - \theta s$

Stress response function $\sigma_{ij} = \hat{\sigma}_{ij}(\theta, \rho, D_{ij}) = -\hat{\pi}_{eq}(\rho, \theta)\delta_{ij} + \hat{\sigma}_{ij}^{vis}(\rho, \theta, D_{ij}) \qquad \hat{\sigma}_{ij}^{vis}(\rho, \theta, 0) = 0$

Heat flux response function $q_i = \hat{q}_i\left(\theta, \rho, \dfrac{\partial \theta}{\partial y_i}, D_{ij}\right)$

$$\hat{\sigma}_{ij}^{vis} = f_1 \mathbf{I} + f_2 \mathbf{D} + f_3 \mathbf{D}^2 \qquad (f_i \text{ are functions of invariants of } \mathbf{D}) \qquad \sigma_{ij}^{vis}(\rho, \theta, D_{ij}) D_{ij} \geq 0 \qquad q_i\left(\rho, \theta, \frac{\partial \theta}{\partial y_i}\right)\frac{\partial \theta}{\partial y_i} \geq 0$$

In addition, the constitutive relations must satisfy

$$\hat{\pi}_{eq} = \rho^2 \frac{\partial \hat{\psi}}{\partial \rho} \qquad \hat{s} = -\frac{\partial \psi}{\partial \theta}$$

$$\frac{\partial \hat{\pi}_{eq}}{\partial \theta} = -\rho^2 \frac{\partial \hat{s}}{\partial \rho} \qquad \hat{\pi}_{eq} = \theta \frac{\partial \hat{\pi}_{eq}}{\partial \theta} + \rho^2 \frac{\partial \hat{\varepsilon}}{\partial \rho}$$

where $c_v(\theta, \rho) = \dfrac{\partial \hat{\varepsilon}}{\partial \theta}$ (Specific heat capacity)

$$c_v = -\theta \frac{\partial^2 \hat{\psi}}{\partial \theta^2} \qquad \frac{\partial c_v}{\partial \rho} = -\frac{\theta}{\rho^2} \frac{\partial^2 \hat{\pi}_{eq}}{\partial \theta^2}$$

Review

Properties of fluids

- No natural reference configuration: Hence constitutive functions can only depend on deformation measures that are independent of reference configuration

$$\rho \qquad \theta \qquad L_{ij} = \frac{\partial v_i}{\partial y_j} \qquad D_{ij} = (L_{ij} + L_{ji})/2 \quad W_{ij} = (L_{ij} - L_{ji})/2$$

Consequences of material frame indifference (objectivity)

Consider $\psi = \hat{\psi}(\rho, \theta, \mathbf{L})$

Observers in all frames in relative rotation must see the same Helmholtz free energy. This means that

$$\psi = \hat{\psi}(\rho, \theta, \mathbf{D})$$

and must satisfy $\hat{\psi}(\rho, \theta, \mathbf{D}) = \hat{\psi}(\rho, \theta, \mathbf{Q}\mathbf{D}\mathbf{Q}^T)$ for all proper orthogonal $\mathbf{Q}$

Same argument applies to all constitutive functions (can only depend on **D**), eg

$$\sigma_{ij} = \hat{\sigma}_{ij}(\theta, \rho, D_{ij}) \qquad q_i = \hat{q}_i\left(\theta, \rho, \frac{\partial \theta}{\partial y_i}, D_{ij}\right)$$

Decomposition $\hat{\sigma}(\rho, \theta, D) = -\Pi_{eq}(\rho,\theta) I + \hat{\sigma}^{vis}(\rho,\theta,D)$

where $\hat{\sigma}^{vis}(\rho, \theta, D=0) = 0$

follows from requirement that @ rest shear stress is zero

Second Law of Thermodynamics

$$\hat{\sigma} : D - \frac{1}{\theta}\hat{q}\cdot\nabla_y\theta - \rho\left(\frac{\partial\hat{\psi}}{\partial t} + s\frac{\partial\theta}{\partial t}\right) \geq 0 \qquad (1)$$

Note $$\frac{\partial\hat{\psi}}{\partial t} = \frac{\partial\hat{\psi}}{\partial\rho}\dot{\rho} + \frac{\partial\hat{\psi}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\psi}}{\partial D}\dot{D} \qquad (2)$$

$$\hat{\sigma} : D = -\Pi_{eq}\, tr(D) + \hat{\sigma}^{vis} : D \qquad (3)$$

Mass conservation $\Rightarrow$ $$\dot{\rho} + \rho\, tr(D) = 0 \qquad (4)$$

Use (1)-(4)

$$\left(\rho^2 \frac{\partial \hat{\psi}}{\partial \rho} - \Pi_{eq}\right) tr(D) + \hat{\sigma}^{vis} : D - \frac{1}{\theta}\hat{q} \cdot \nabla_y \theta - \rho\left(s + \frac{\partial \hat{\psi}}{\partial \theta}\right)\dot{\theta} - \rho \frac{\partial \hat{\psi}}{\partial D}\dot{D} \geqslant 0$$

Now suppose $D = 0$, $\dot{\theta} = 0$ $\nabla_y \theta = 0$ $\dot{D} \neq 0$

Hence $\frac{\partial \hat{\psi}}{\partial D} = 0 \Rightarrow$ $\boxed{\hat{\psi} \text{ cannot depend on } D}$

Next let $\dot{\theta} = 0$, $\nabla_y \theta = 0$ $D = \alpha B$ $\alpha \geqslant 0$

B an arbitrary tensor

$$\Rightarrow \left(\rho^2 \frac{\partial \hat{\psi}}{\partial \rho} - \Pi_{eq}\right) \cancel{\alpha}\, tr(B) + \hat{\sigma}^{vis}(\rho, \theta, \alpha B) : \cancel{\alpha} D \geqslant 0$$

Now let $\alpha \to 0$

Hence $\boxed{\rho^2 \frac{\partial \hat{\psi}}{\partial \rho} = \Pi_{eq}}$

Next $\dot{\theta} \neq 0$, everything else 0 $\Rightarrow$ $\boxed{\hat{s} + \frac{\partial \hat{\psi}}{\partial \theta} = 0}$

Finally $\boxed{\hat{\sigma}^{vis} : D \geq 0}$ $\boxed{-\frac{1}{\theta} \hat{q} \cdot \nabla_y \theta \geq 0}$

Remaining identities follow from these: eg

$$\Pi_{eq} = \rho^2 \frac{\partial \hat{\psi}}{\partial \rho} \qquad \hat{s} = -\frac{\partial \hat{\psi}}{\partial \theta}$$

$$\Rightarrow \boxed{\frac{\partial \Pi_{eq}}{\partial \theta} = \rho^2 \frac{\partial^2 \hat{\psi}}{\partial \rho \partial \theta} = -\rho^2 \frac{\partial \hat{s}}{\partial \rho}}$$

similarly $\quad \Pi_{eq} = \rho^2 \dfrac{\partial \hat{\psi}}{\partial \rho} = \rho^2 \dfrac{\partial(\hat{\varepsilon} - \theta \hat{s})}{\partial \rho} = \rho^2 \dfrac{\partial \varepsilon}{\partial \rho} - \theta \rho^2 \dfrac{\partial \hat{s}}{\partial \rho}$

$$\Rightarrow \Pi_{eq} = \rho^2 \frac{\partial \varepsilon}{\partial \rho} + \theta \frac{\partial \Pi_{eq}}{\partial \theta}$$

Now consider stress response function $\hat{\sigma}(\rho, \theta, D)$

For material frame indifference we must have

$$Q \hat{\sigma}(\rho, \theta, D) Q^T = \hat{\sigma}(\rho, \theta, Q D Q^T) \quad \forall Q : Q Q^T = I$$

Functions with this property are called "isotropic functions"

Representation theorem for isotropic tensors

Most general form of an isotropic function is

$$\hat{\sigma} = f_1 I + f_2 D + f_3 D^2$$

where f_i are functions of the invariants of D

Note Π_{eq} is independent of D $\quad \Pi_{eq} I = Q \Pi_{eq} I Q^T$
$= \Pi_{eq} I$

Hence $\hat{\sigma}^{vis} = f_1 I + f_2 D + f_3 D^2$

Can easily show $\hat{\sigma}^{vis}(QDQ^T) = Q \hat{\sigma}^{vis}(D) Q^T$

$$\hat{\sigma}^{vis}(QDQ^T) = f_1 \underbrace{I}_{QQ^T} + f_2 QDQ^T + f_3 QD\cancel{Q^T}\cancel{Q}DQ^T$$

$$= Q[f_1 I + f_2 D + f_3 D^2] Q^T$$
$$= Q \hat{\sigma}(D) Q^T$$

Examples of specific constitutive models for fluids

(1) Ideal Fluid (Euler fluid)

$$\psi = \hat{\psi}(\rho), \quad \hat{\sigma}^{vis} = 0 \qquad \text{(inviscid)}$$

(2) Ideal Gas

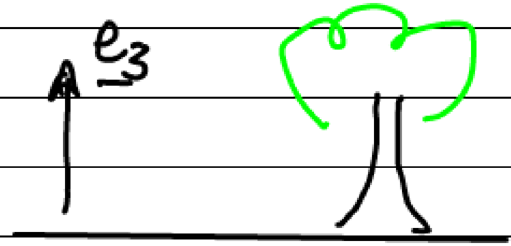
$$\hat{\psi} = c_v \theta - \theta(c_v \log\theta - R\log\rho + s_0) \qquad \hat{\sigma}_{ij}^{vis} = 0$$

c_v - specific heat capacity (constant)

R - individual gas constant $\quad R = \frac{R_u}{m} \quad R_u = 8.314 \frac{kJ}{mol K}$

$C_p = c_v + R \qquad \gamma = \frac{c_p}{c_v}$ $\qquad m$ - molar mass

Example using ideal gas law: Estimate pressure / density of air above earth's surface



Static equilibrium, const temp

$$\sigma_{ij} = -\Pi_{eq}\,\delta_{ij}$$

$$\Pi_{eq} = \rho^2 \frac{\partial \hat{\psi}}{\partial \rho} = \rho R\theta$$

Linear momentum $\dfrac{\partial \sigma_{33}}{\partial y_3} - \rho g = 0$ $\quad \dfrac{\partial \sigma_{11}}{\partial y_1} = \dfrac{\partial \sigma_{22}}{\partial y_2} = 0$

$$\Rightarrow R\theta \frac{\partial \rho}{\partial y_3} + \rho g = 0$$

$\rho_0 R\theta$

$$\Rightarrow \rho = \rho_0 \exp\left(-y_3\, g/R\theta\right) \qquad p = p_0 \exp\left(-y_3\, g/R\theta\right)$$