Review – Examples of constitutive models for fluids

- Ideal Fluid (Eulerean fluid)

  Free energy $\psi = \hat{\psi}(\rho, \theta)$ Inviscid $\hat{\sigma}_{ij}^{vis} = 0$

- Ideal Gas

  Free energy $\hat{\psi} = c_v \theta - \theta(c_v \log(\theta) - R \log(\rho) - s_0)$

  Inviscid $\hat{\sigma}_{ij}^{vis} = 0$

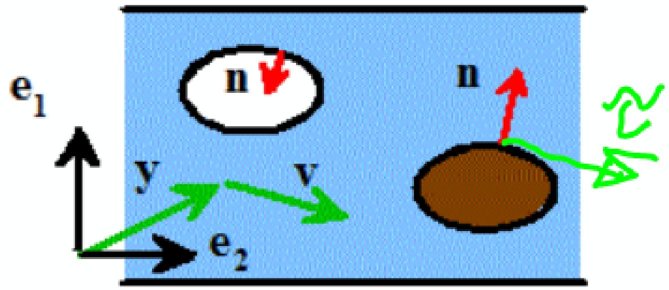
Compressible linear viscous fluid

$$\hat{\sigma} = -\Pi_{eq} I + \kappa \, tr(D) + 2\eta \left(D - \frac{1}{3} tr(D) I \right) \quad *$$

$\kappa > 0$ - Bulk viscosity

$\eta > 0$ - shear viscosity

6.) Fluid Mechanics



Goal: Calculate
- velocity $\underline{v}$
- Density ρ
- stress σ
- Temperature θ (sometimes)

Given: Constitutive model $\hat{\sigma}$, ψ $\hat{q}$: – most common htrf

$\hat{q} = q = -k \nabla_y \theta$

Initial conditions:

ρ, $\underline{v}$, θ @ $t = 0$

Boundary conditions

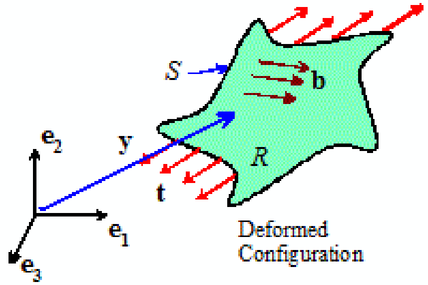
(1) $\underline{v} \cdot \underline{n}$ continuous @ all interfaces

(2) $\underline{n} \cdot \sigma$ " " " "

(3) Tangential law No slip $\underline{v}$ continuous

Frictionless $\Rightarrow$ $n \cdot \sigma \cdot \tau = 0$

Summary of equations relevant to fluids



Kinematics

- Only need variables that don't depend on ref. config

$$L_{ij} = \frac{\partial v_i}{\partial y_j} \quad L_{ij} = D_{ij} + W_{ij} \qquad D_{ij} = (L_{ij} + L_{ji})/2 \quad W_{ij} = (L_{ij} - L_{ji})/2$$

$$a_i = \left.\frac{\partial v_i}{\partial t}\right|_{x_k=const} = \frac{\partial v_i}{\partial y_k}\frac{\partial y_k}{\partial t} + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} = L_{ik}v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} = (D_{ik} + W_{ik})v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const}$$

$$= \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + 2W_{ik}v_k = \left.\frac{\partial v_i}{\partial t}\right|_{y_k=const} + \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + \epsilon_{ijk}\,\omega_j v_k$$

$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial y_j} = -\epsilon_{ijk} W_{ij}$$

Conservation Laws

Mass $\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho D_{kk} = 0$ or $\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0$

Energy $\rho\left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \sigma_{ij}D_{ij} - \frac{\partial q_i}{\partial y_i} + q$

BLM $\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho\left(\frac{\partial v_i}{\partial y_k}v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const}\right) \qquad \sigma_{ij} = \sigma_{ji}$

2nd Law $\sigma_{ij}D_{ij} - \frac{1}{\theta}q_i\frac{\partial \theta}{\partial y_i} - \rho\left(\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t}\right) \geq 0$

Constitutive Equations

$$\psi = \hat{\psi}(\rho, \theta) = \varepsilon - \theta s$$

$$\sigma_{ij} = \hat{\sigma}_{ij}(\theta, \rho, D_{ij}) = -\hat{\pi}_{eq}(\rho, \theta)\delta_{ij} + \hat{\sigma}_{ij}^{vis}(\rho, \theta, D_{ij}) \qquad \hat{\pi}_{eq} = \rho^2\frac{\partial \hat{\psi}}{\partial \rho}$$

$\hat{\sigma}_{ij}^{vis} = f_1\mathbf{I} + f_2\mathbf{D} + f_3\mathbf{D}^2$ (f_i are functions of invariants of $\mathbf{D}$) $\hat{\sigma}_{ij}^{vis}(\rho, \theta, 0) = 0$

$$q_i = \hat{q}_i\left(\theta, \rho, \frac{\partial \theta}{\partial y_i}, D_{ij}\right) = -k\frac{\partial \theta}{\partial y_i}$$

Re-formulation of governing equations for fluids

Mass conservation : $\left.\frac{\partial \rho}{\partial t}\right|_y + \nabla_y \cdot (\rho \underline{v}) = 0 \Leftarrow$ gives ρ

Incompressible fluid $\rho = \text{constant}$ $\nabla_y \cdot \underline{v} = 0$

Navier-Stokes equation (linear viscous fluid)

$$\boxed{-\frac{\partial p}{\partial y_i} + 2\frac{\partial}{\partial y_j}\left(\eta\left(D_{ij} - \frac{D_{kk}\delta_{ij}}{3}\right)\right) + \rho b_i = \rho a_i}$$

$$\boxed{p = \Pi_{eq} - \kappa D_{kk}}$$

Derivation : Substitute (*) into LMB ; simplify

Special Case ① Assume η, κ constant

$$-\frac{1}{\rho}\frac{\partial \Pi_{eq}}{\partial y_i} + \underbrace{\frac{\eta}{\rho}}_{\nu\text{ - kinematic viscosity}}\frac{\partial^2 v_i}{\partial y_j \partial y_j} + \left(\frac{\kappa}{\rho} - \frac{2}{3}\frac{\eta}{\rho}\right)\frac{\partial^2 v_j}{\partial y_j \partial y_i} + b_i = a_i$$

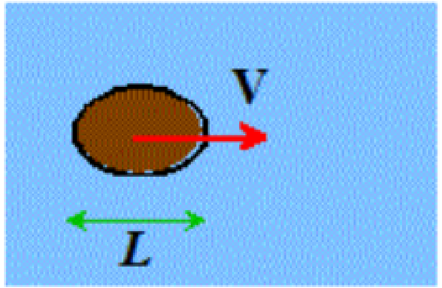
Special case ② : $\eta, \kappa \to 0$ Ideal fluid

$$-\frac{1}{\rho}\frac{\partial \Pi_{eq}}{\partial y_i} + b_i = a_i \qquad \text{Euler equation}$$

Special case ③ Incompressible flows :

$$-\frac{1}{\rho}\frac{\partial p}{\partial y_i} + \frac{\eta}{\rho}\frac{\partial^2 v_i}{\partial y_j \partial y_j} + b_i = a_i$$

Normalized Navier-Stokes equation



Define: L – Characteristic length
$\bar{V}$ – Characteristic velocity
f – Characteristic frequency
Δp – Characteristic pressure

Normalize variables $y_i = \hat{y}_i L$ $\quad v_i = \hat{v}_i \bar{V}$
$p = \Delta p \, \hat{p}$ $\quad b_i = \hat{b}_i g$ $\quad t = \hat{t}/f$

* Reynolds number $Re = \rho \bar{V} / \eta$ $\qquad \rho \bar{V} / k$

* Euler number $Eu = \Delta p / \rho \bar{V}^2$

* Froude number $Fr = \bar{V} / \sqrt{gL}$

* Strouhal number $St = f L / V$

Normalized N-S (incompressible)

$$-Eu \frac{\partial \hat{p}}{\partial \hat{y}_i} + \frac{1}{Re} \frac{\partial^2 \hat{v}_i}{\partial \hat{y}_j \partial \hat{y}_j} + \frac{1}{Fr^2} \hat{b}_i = St \left.\frac{\partial \hat{v}_i}{\partial \hat{t}}\right|_{\underline{x}}$$

Additional field equations for limiting cases

Ideal (Inviscid) flows $\quad Re \to \infty$

LMB (Euler) $\quad \dfrac{1}{\rho} \dfrac{\partial \Pi_{eq}}{\partial y_i} + b_i = a_i$

Vorticity Transport equation $\quad$ Define $\underline{\omega} = \nabla_y \times \underline{v}$

$$\nabla_y \times \underline{b} + D\underline{\omega} - tr(D)\,\underline{\omega} = \left.\frac{\partial \underline{\omega}}{\partial t}\right|_{\underline{x}}$$

Proof: (1) $\nabla_y \times (\nabla_y \phi) = 0$ for any ϕ

(2) Recall $\nabla_y \times \underline{a} = \left.\dfrac{\partial \underline{\omega}}{\partial t}\right|_{\underline{x}} - D\underline{\omega} + tr(D)\,\underline{\omega}$

Take $\nabla_y \times$ (Euler) $\Rightarrow$ QED

Irrotational Flow

Suppose $\underline{b} = \nabla_y \psi$ for some potential ψ

Suppose $\underline{\omega} = 0$ @ $t = 0$

Then $\dfrac{\partial \underline{\omega}}{\partial t} = 0 \Rightarrow \underline{\omega} = 0 \Leftarrow$ Irrotational Flow