Review – Field Equations for Ideal Fluids

- Simplified form of Navier-Stokes (Euler equation)

$$-\frac{1}{\rho}\frac{\partial \pi_{eq}}{\partial y_i}+b_i=\left.\frac{\partial v_i}{\partial t}\right|_{y_k=const}+\frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k)+\in_{ijk}\ \omega_j v_k$$

- Vorticity Transport Equation

$$\in_{ijk}\frac{\partial}{\partial x_j}(b_k)+D_{ij}\omega_j-\frac{\partial v_k}{\partial y_k}\omega_i=\left.\frac{\partial \omega_i}{\partial t}\right|_{\mathbf{x}=const}$$

- Irrotational Flow: Assume $\omega_i=0$ at time *t=0*, and that $\in_{ijk}\frac{\partial}{\partial x_j}(b_k)=0$
  Then $\omega_i=0$ for all time – flow is said to be **irrotational**.

Bernoulli Equation:

Assumptions:

(1) Ideal fluid – $\psi(\rho)$ $\quad \pi_{eq} = \rho^2 \dfrac{\partial \psi}{\partial \rho}$

(2) Body force can be derived from a potential $\underline{b} = -\nabla_y \phi$

(3) Steady flow: $\left.\dfrac{\partial \underline{v}}{\partial t}\right|_y = 0$

Define streamline

Curves that are everywhere tangent to velocity

(Path traced by material particles)

Streamline is a path traced out by a massless particle moving with the flow.

Velocity is tangent to streamline at every point.

Mass does not cross streamlines.

Define $\boxed{H = \psi + \frac{\Pi_{eq}}{\rho} + \frac{1}{2}|\underline{v}|^2 + \phi}$

Bernoulli : $\boxed{H = \text{constant along streamlines}}$

For irrotational flow $H = \text{const}$ everywhere

Proof : (1) $\frac{1}{\rho}\frac{\partial \Pi_{eq}}{\partial y_i} = \frac{\partial}{y_i}\left(\psi + \frac{\Pi_{eq}}{\rho}\right)$

To see this $\frac{\partial \psi}{\partial y_i} = \frac{\partial \psi}{\partial \rho}\frac{\partial \rho}{\partial y_i}$ $\qquad \frac{\partial}{\partial y_i}\left(\frac{\Pi_{eq}}{\rho}\right) = \frac{1}{\rho}\frac{\partial \Pi_{e}}{\partial y_i} - \frac{\Pi_{eq}}{\rho^2}\frac{\partial \rho}{\partial y_i}$

Recall $\Pi_{eq} = \rho^2 \frac{\partial \psi}{\partial \rho} \Rightarrow (1)$

Now N-S eq can be re-written

$$-\frac{\partial}{\partial y_i}\left(\psi + \frac{\Pi_{eq}}{\rho}\right) - \frac{\partial \phi}{\partial y_i} = \frac{\partial}{\partial y_i}\frac{1}{2}|\underline{V}|^2 + \underline{\omega} \times \underline{V}$$

$$\Rightarrow \quad -\frac{\partial H}{\partial y_i} = \underline{\omega} \times \underline{V} \qquad (2)$$

Now consider curve tangent to V

Dot (2) on both sides with tangent

$$-\frac{dH}{ds} = 0 \quad \Rightarrow \quad H = \text{const along curve}$$

Irrotational $\Rightarrow$ $H = \text{const}$

Potential Flow

Assumptions : (1) Irrotational
(2) Incompressible
(3) Inviscid

(1) $\Rightarrow \quad \nabla_y \times \underline{V} = 0 \quad \Leftrightarrow \quad \underline{V} = \nabla_y \Omega \quad \Omega(y,t)$

Ω = scalar function

(2) $\Rightarrow \quad \nabla_y \cdot \underline{V} = 0 \quad \Rightarrow \quad \nabla_y \cdot \nabla_y \Omega = \boxed{\nabla_y^2 \Omega = 0}$

This equation (with boundary conditions) is the governing eq for $\underline{V}$

Bernoulli gives pressure

$$\boxed{\frac{p}{\rho} + \frac{1}{2} |\underline{V}|^2 + \phi + \frac{\partial \Omega}{\partial t} = f(t)}$$

Incompressible Viscous Flow

N-S :

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y_i} + \frac{\eta}{\rho}\frac{\partial v_i}{\partial y_j \partial y_j} + b_i = \frac{\partial v_i}{\partial t}\Big|_y + \frac{\partial}{\partial y_i}\left(\frac{1}{2}|\underline{v}|^2\right) + \underline{\omega} \times \underline{v}}$$

Taking curl gives vorticity transport eq

$$\boxed{\nabla_y \times \underline{b} + \frac{\eta}{\rho}\nabla_y^2 \underline{\omega} + D\underline{\omega} = \frac{\partial \underline{\omega}}{\partial t}\Big|_{\underline{x}}}$$

Viscous flows need not be irrotational

Stokes flows :

Assume $|V|^2 \leq |\nabla_y^2 V|$

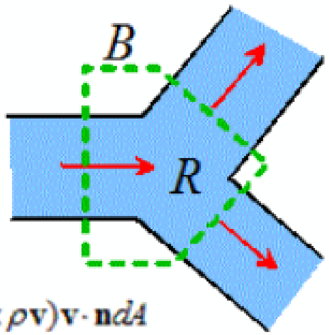
Then can neglect nonlinear terms in velocity

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y_i} + \frac{\eta}{\rho}\frac{\partial^2 v_i}{\partial y_j \partial y_j} + b_i \approx \left.\frac{\partial v_i}{\partial t}\right|_y}$$

Solutions to fluid mechanics problems

(1) Control vol method
(2) Potential Flow
(3) Compressible ideal flows
(4) Stokes flow

Review: from earlier in notes



Mass Conservation: $\frac{d}{dt}\int_R \rho dV + \int_B \rho \mathbf{v}\cdot\mathbf{n}dA = 0$

Linear Momentum Balance $\int_B \mathbf{n}\cdot\boldsymbol{\sigma}dA + \int_R \rho\mathbf{b}dV = \frac{d}{dt}\int_R \rho\mathbf{v}dV + \int_B (\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}dA$

Angular Momentum Balance $\int_B \mathbf{y}\times(\mathbf{n}\cdot\boldsymbol{\sigma})dA + \int_R \mathbf{y}\times(\rho\mathbf{b})dA = \frac{d}{dt}\int_R \mathbf{y}\times\rho\mathbf{v}dV + \int_B (\mathbf{y}\times\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}dA$

Mechanical Power Balance $\int_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v}dA + \int_R \rho\mathbf{b}\cdot\mathbf{v}dV = \int_R \boldsymbol{\sigma}:\mathbf{D}dV + \frac{d}{dt}\int_R \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})dV + \int_B \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})\mathbf{v}\cdot\mathbf{n}dA$

First law of thermodynamics

$$\int_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v}dA + \int_R \rho\mathbf{b}\cdot\mathbf{v}dV - \int_B \mathbf{q}\cdot\mathbf{n}dA + \int_V q dV = \frac{d}{dt}\int_R \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)dV + \int_B \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\cdot\mathbf{n}dA$$

Second law of thermodynamics

$$\frac{d}{dt}\int_R \rho s dV + \int_B \rho s(\mathbf{v}\cdot\mathbf{n})dA + \int_B \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}dA - \int_R \frac{q}{\theta}dV \geq 0$$

Can use these to solve some problems directly

Assume ideal fluid

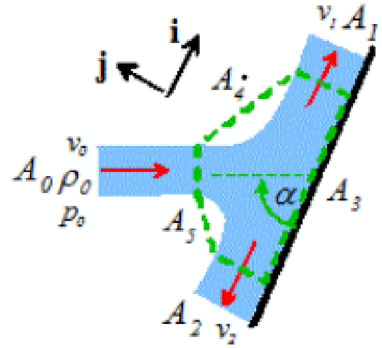
- no shear stress on wall

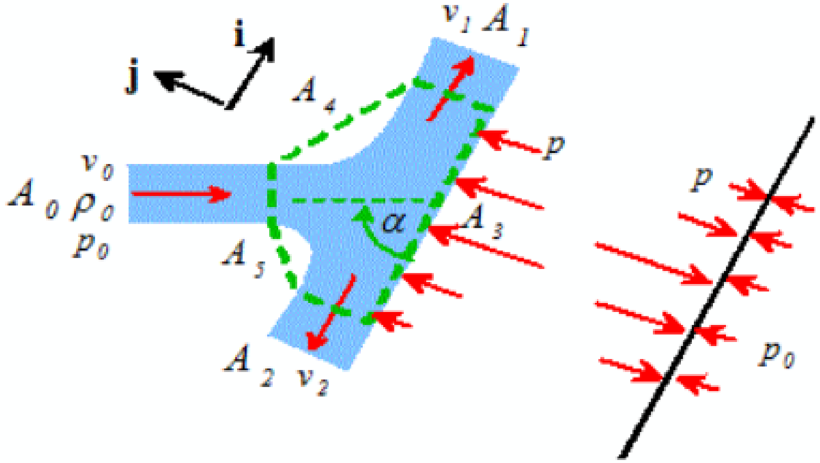
Force on wall is

$$-\int_{A_3} (p - p_0)\, dA\, \mathbf{j} = \underline{F}$$

Apply LMB to control vol shown

Example: Fluid jet hits an inclined wall. Calculate resultant force on the wall.





Linear Momentum: $\int_A \underline{n} \cdot \sigma \, dA = \int_A (\rho \underline{v}) \, \underline{v} \cdot \underline{n} \, dA \qquad (1)$

On A_1, A_2, A_4, A_5 $\quad \underbrace{\sigma = -p_0 I}_{\text{Atmospheric}}$ $\qquad$ on A_3 $\sigma = -pI$

on A_2, A_1 $\quad \underline{v} = \pm v \underline{i}$ $\qquad$ on A_3 $\underline{v} \cdot \underline{n} = 0$

on A_0 $\quad \underline{v} = v_0 \cos\alpha \underline{i} - v_0 \sin\alpha \, j \qquad \underline{n} = j$

Hence (1) reduces to

$$\int_{A_1+A_2+A_4+A_5} -p_0 \underline{n} + \int_{A_3} p j \, dA = \int_{A_0} \rho V_0 (\cos\alpha \underline{i} - \sin\alpha j)(-V_0 \sin\alpha)$$

$$+ \int_{A_2} \rho v_2 \underline{i} \; v_2 \, dA + \int_{A_1} \rho v_1 \underline{i} \; v_1 \, dA$$

Note $\int_{A_1+A_2+A_4+A_5} -p_0 \underline{n} = \int_{A_1+A_2+A_3+A_4+A_5} -p_0 \underline{n}\, dA + \int_{A_3} p_0 \underline{n}\, dA$

$\underline{n} = -\underline{j}$ on A_3

Note also $\int_{A_1+A_2+A_3+A_4+A_5} p_0 \underline{n}\, dA = 0$ (no force from uniform pressure on closed surface)

Hence $\underbrace{\int_{A_3} (p-p_0)\underline{j}\, dA}_{= -\underline{F}} = \rho v_0^2 \sin\alpha A_0 \underline{j} \underbrace{- \rho v_0^2 \cos\alpha A_0 \underline{i} + \int_{A_2} \rho v_2 \underline{i}\, v_2\, dA + \int_{A_1} \rho v_1 \underline{i}\, v_1\, dA}_{= 0}$

$\underline{F} = -\rho v_0^2 \sin\alpha A_0 \underline{j}$

Assumptions:

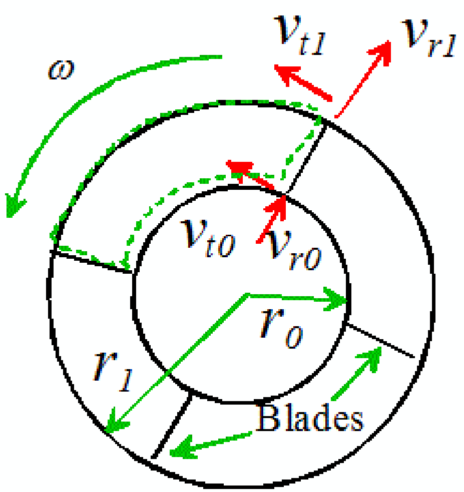
Ideal, incompressible fluid

Infinite number of vanes

Pump rotates with angular speed ω

The mass flow rate is μ

Estimate the torque on the shaft



volute centrifugal pump

http://youtu.be/V3aPHmZ97yM