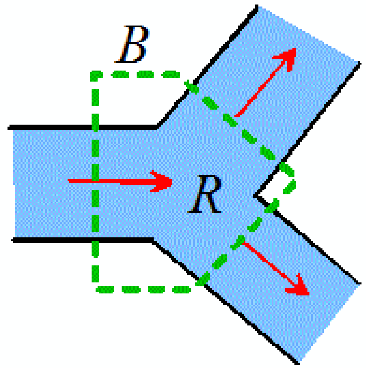
Review – Analytical solutions for fluids

- Control volume methods
- Potential flow
- Stokes Flow
- Compressible ideal flows



- **Mass Conservation:** $\frac{d}{dt}\int_R \rho dV + \int_B \rho \mathbf{v}\cdot\mathbf{n} dA = 0$
- **Linear Momentum Balance** $\int_B \mathbf{n}\cdot\boldsymbol{\sigma} dA + \int_R \rho\mathbf{b} dV = \frac{d}{dt}\int_R \rho\mathbf{v} dV + \int_B (\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n} dA$
- **Angular Momentum Balance** $\int_B \mathbf{y}\times(\mathbf{n}\cdot\boldsymbol{\sigma}) dA + \int_R \mathbf{y}\times(\rho\mathbf{b}) dA = \frac{d}{dt}\int_R \mathbf{y}\times\rho\mathbf{v} dV + \int_B (\mathbf{y}\times\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n} dA$
- **Mechanical Power Balance**

$$\int_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v} dA + \int_R \rho\mathbf{b}\cdot\mathbf{v} dV = \int_R \boldsymbol{\sigma}:\mathbf{D} dV + \frac{d}{dt}\int_R \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v}) dV + \int_B \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})\mathbf{v}\cdot\mathbf{n} dA$$

- **First law of thermodynamics**

$$\int_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v} dA + \int_R \rho\mathbf{b}\cdot\mathbf{v} dV - \int_B \mathbf{q}\cdot\mathbf{n} dA + \int_V q dV = \frac{d}{dt}\int_R \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) dV + \int_B \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\cdot\mathbf{n} dA$$

Assumptions : Ideal incompressible flow

Sufficiently large # of vanes to ensure that velocity can be approximated as

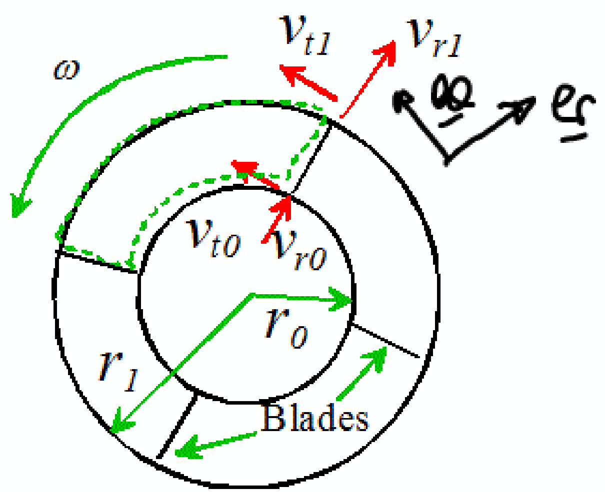
$$\underline{v} = v_r \underline{e_r} + v_t \underline{e_\theta}$$

$$v_t = r\omega \text{ throughout}$$

Pump rotates with angular speed ω

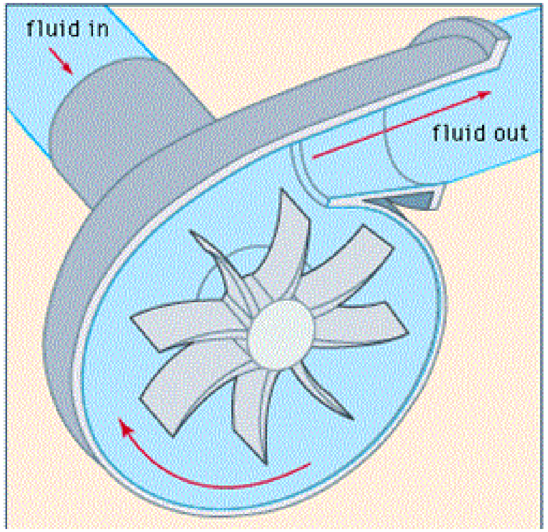
The mass flow rate is μ

Estimate the torque on the shaft



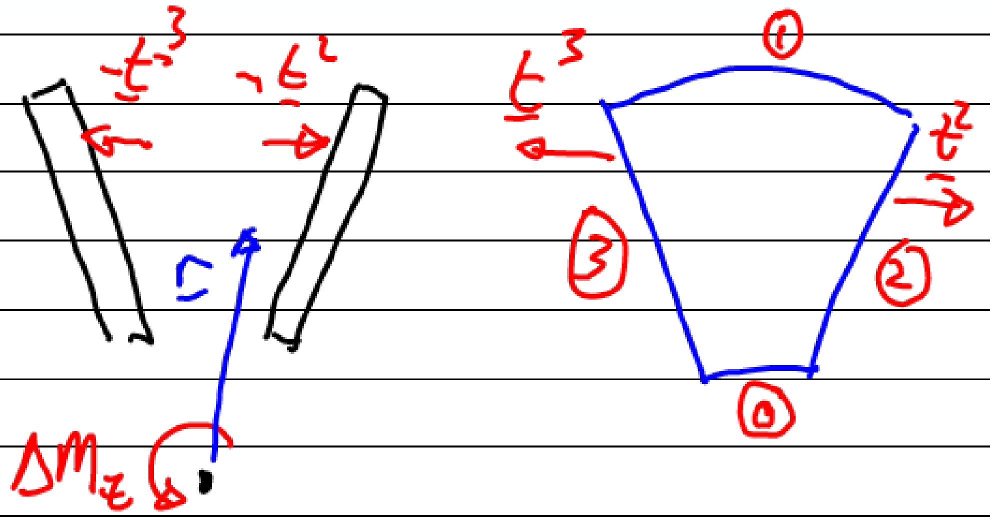
volute centrifugal pump

http://youtu.be/V3aPHmZ97yM



Equilibrium of vanes

$$\underline{e_z} \Delta M_z + \int_{A_3+A_2} \underline{r} \times (-\underline{t}) = \underline{0}$$



Now consider BAM : $\int_B \underline{y} \times (\underline{n} \cdot \underline{\sigma})\, dA = \int_B (\underline{y} \times \rho \underline{v})(\underline{v} \cdot \underline{n})\, dA$

Note stress is hydrostatic $\sigma = -pI$
Tractions on all surfaces act normal to surfaces

Note $\underline{r} \times (\underline{n} \cdot \sigma) = 0$ on A_0, A_1

Hence $\int_B \underline{y} \times (\underline{n} \cdot \sigma) = \int_{A_3 + A_2} \underline{r} \times \underline{t}\, dA$

Note $\underline{n} = -\underline{e_r}$ on A_0 $\quad$ $\underline{n} = \underline{e_r}$ on A_1

$\underline{n} = -\underline{e_\theta}$ on A_2 $\quad$ $\underline{n} = \underline{e_\theta}$ on A_3

Hence

$$\int \underline{y} \times (\rho \underline{v})\,(\underline{v}\cdot n)\,dA = \rho \int_{A_0} r_0 \underline{e_r} \times (v_r \underline{e_r} + v_t \underline{e_\theta})\,(-v_{r0})\,dA$$

$$\rho \int_{A_1} r_1 \underline{e_r} \times (v_r \underline{e_r} + v_t \underline{e_\theta})\,(v_{r1})\,dA$$

$$\rho \int_{A_2} r\, \underline{e_r} \times (v_r \underline{e_r} + \cancel{v_t} \underline{e_\theta})\,(-v_t)\,dA$$

$$\rho \int_{A_3} r\, \underline{e_r} \times (v_r \underline{e_r} \cancel{+ v_t} \underline{e_\theta})\,(v_t)\,dA$$

$$= \cancel{\rho} r_0\, r_0 \omega\, \overbrace{(-\cancel{v_{r0}})\, \cancel{r_0} \cancel{\Delta\theta}}^{-\mu\Delta\theta/2\pi} + \cancel{\rho} r_1^2 \omega\, \overbrace{v_{r1}\, \cancel{r_1} \cancel{\Delta\theta}}^{\mu\Delta\theta/2\pi}$$

Mass conservation : $\mu \dfrac{\Delta\theta}{2\pi} \equiv \rho v_{r0}\, r_0 \Delta\theta = \rho v_{r1} r_1 \Delta\theta$

Hence total moment $M_z = \Delta M_z \frac{2\pi}{\Delta\theta} = \mu\omega(r_1^2 - r_0^2)$

Potential Flow

Review : Inviscid ; incompressible ; irrotational , conservative body forces

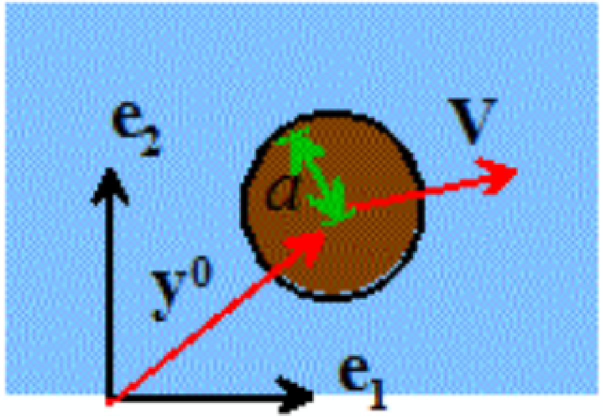
Solution Method: Find Ω satisfying $\nabla_y^2 \Omega = 0$

plus $\nabla_y \Omega \cdot \underline{n} = V_n^*$ on surfaces with velocity V_n^*

Then $\underline{V} = \nabla_y \Omega$

and pressure follows from Bernoulli

Example: Flow past a moving rigid sphere



Potential $\Omega = -\dfrac{a^3 V_k}{2R^3}(y_k - y_k^0)$

$R = |y - y^0|$

Hence $v_i = \dfrac{\partial \Omega}{\partial y_i} = -\dfrac{a^3 V_i}{2R^3} + \dfrac{3a^3 V_k (y_k - y_k^0)(y_i - y_i^0)}{2R^5}$

Check: $\dfrac{\partial v_i}{\partial y_i} = \nabla_y^2 \Omega = +\dfrac{3a^3 V_i (y_i - y_i^0)}{2R^5}$

$- \dfrac{15 a^3 V_k (y_k - y_k^0)}{2R^5}$

$$+ \frac{9a^3 \bar{V}_i (y_i - y_i^0)}{2R^5}$$

$$+ \frac{3a^3 \bar{V}_i (y_i - y_i^0)}{2R^5} = 0 \checkmark !$$

Boundary Condition $v_i n_i = \bar{V}_i n_i$ on $R = a$

$$v_i n_i = -v_i \left(\frac{y_i - y_i^0}{a} \right) = \bar{V}_i \left(\frac{y_i - y_i^0}{a} \right) \quad \text{on } R = a \checkmark$$

Pressure follows from Bernoulli

$$\frac{p}{\rho} + \frac{1}{2} |\underline{v}|^2 = \frac{p_0}{\rho}$$

$$\frac{p}{\rho}+\frac{1}{2}\left(\frac{a^3V_i}{2r^3}\frac{a^3V_i}{2r^3}-3\frac{a^3V_k(y_k-V_kt)}{r^4}\frac{(y_i-V_it)}{r}\frac{a^3V_i}{2r^3}+3\frac{a^3V_k(y_k-V_kt)}{2r^4}3\frac{a^3V_n(y_n-V_nt)}{2r^4}\right)$$

$$+\frac{a^3V_iV_i}{2r^3}-3\frac{a^3V_k(y_k-V_kt)}{2r^4}\frac{(y_i-V_it)V_i}{r}=\frac{p_0}{\rho}$$

$$\frac{p}{\rho}=\frac{p_0}{\rho}-\frac{a^3V_iV_i}{2r^3}-\frac{a^6V_iV_i}{8r^6}+9\frac{a^3V_k(y_k-V_kt)}{8r^4}\frac{(y_i-V_it)V_i}{r}$$

Tractions on sphere $t_j = -p\,\delta_{ij}\left(\frac{y_i-y_i^0}{a}\right)$

Resultant force on sphere $F_i = \int_A t_i\,dA$

Integral vanishes by symmetry

Resultant force is always zero.

Effective mass of sphere in an ideal fluid

$$\underline{F} \cdot \underline{V} = \frac{d}{dt}\left(KE_{sphere} + KE_{fluid}\right)$$

Hence can get apparent mass by calculating KE of fluid

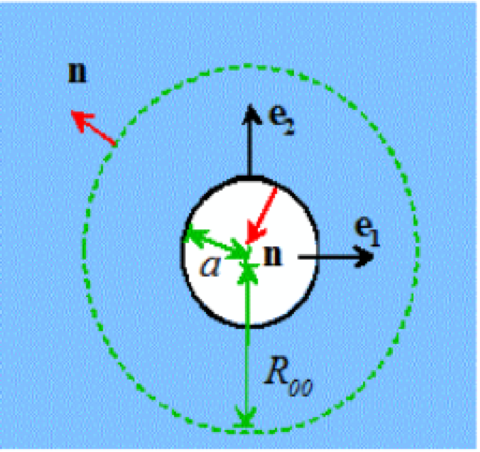
$$KE_{fluid} = \int_V \tfrac{1}{2}\rho|\underline{v}|^2 dV$$

$$= \int_V \tfrac{1}{2}\rho \frac{\partial \Omega}{\partial y_i} v_i \, dV$$

$=0$ (incompressible?)

$$= \int_V \left(\tfrac{1}{2}\rho \frac{\partial}{\partial y_i}(\Omega v_i) - \tfrac{1}{2}\rho\, \Omega \cancel{\frac{\partial v_i}{\partial y_i}}\right) dV$$

Div thm: $KE_{fluid} = \frac{1}{2}\rho \int_A \Omega v_i n_i \, dA$



$$= \frac{1}{2}\rho \int_{S_1} \bar{V}_i \left(-\frac{y_i}{a}\right)\left(\frac{-a^3}{2a^3}\bar{V}_k y_k\right) dA$$

$$+ \frac{\rho}{2}\int_{R_\infty} \underbrace{\Omega v_i}_{O(|R|^{-5})} n_i \underbrace{dA}_{O(R^2)} \rightarrow 0 \text{ as } R_\infty \rightarrow \infty$$

Hence $KE_{fluid} = \frac{\rho}{4}\int_{S_1} a\bar{V}_i \bar{V}_k \frac{y_i y_k}{a^2} \, dA$

Consider $\int_{S_1} \frac{y_i y_k}{a^2} \, dA = 0$ for $i \neq k$

For $i = k$ let $i = k = 3$ (same for all $i = 1, 2, 3$)
and use polar coords

$$\int_0^{2\pi}\int_0^{\pi} \frac{a^2 \cos^2\theta}{a^2} \, a^2 \sin\theta \, d\theta \, d\phi = \frac{4}{3}\pi a^2$$

$$KE\ fluid = \frac{\rho}{4} a^3 \frac{4\pi}{3} \bar{V}_i \bar{V}_i = \frac{1}{2} \frac{m_{fluid}}{2} V_i V_i$$

where $m_{fluid} = \frac{4}{3}\pi a^3 \rho$

is mass of fluid displaced by sphere