Review – constitutive models for hyperelastic materials

Goal: develop an exact stress-strain law that describes large deformations of elastomeric materials in the ‘rubbery’ regime

Assumptions:
- Local Action
- Perfectly Reversible
- History independent

\[ \mathbf{C} = \mathbf{F}^T \mathbf{F} \]

We will show that the constitutive law must have the following structure:

Specific internal energy \( \varepsilon = \hat{\varepsilon}(\mathbf{C}, \theta) \)

Specific entropy \( s = \hat{s}(\mathbf{C}, \theta) \)

Specific Helmholtz free energy \( \psi = \hat{\psi}(\mathbf{C}, \theta) \)

Material stress response function \( \Sigma = \hat{\Sigma}(\mathbf{C}, \theta) \)

Material heat flux response function \( \mathbf{Q} = \hat{\mathbf{Q}}(\mathbf{C}, \theta) \)

\[ \Sigma_{ij} = 2 \rho_0 \frac{\partial \hat{\psi}}{\partial C_{ij}} \quad \mathbf{s} = -\frac{\partial \hat{\psi}}{\partial \theta} \]

\[ \frac{\partial \Sigma_{ij}}{\partial \theta} = -2 \rho_0 \frac{\partial \hat{s}}{\partial C_{ij}} \quad -\hat{\mathbf{Q}} \cdot \nabla \theta \geq 0 \]

Notes: Other equivalent forms/relations exist
Other less general forms are also often used
Review

Deformation Gradient
\[ \mathbf{F} = \nabla \mathbf{y} = \mathbf{I} + \mathbf{\nabla u} \quad J = \det(\mathbf{F}) \]

Cauchy-Green Strains
\[ \mathbf{C} = \mathbf{F}^T \mathbf{F} \]
\[ \mathbf{B} = \mathbf{FF}^T \]

Nominal Stress
\[ \mathbf{S} = J \mathbf{F}^{-1} \cdot \mathbf{\sigma} \]

Material Stress
\[ \mathbf{\Sigma} = J \mathbf{F}^{-1} \cdot \mathbf{\sigma} \cdot \mathbf{F}^{-T} \]
\[ \Sigma_{ij} = J F_{ik}^{-1} \sigma_{kl} F_{jl}^{-1} \]

Referential Heat Flux
\[ \mathbf{Q} = J \mathbf{F}^{-1} \cdot \mathbf{q} \]

Dissipation Inequality
\[ \Sigma : \frac{\partial \mathbf{F}}{\partial t} - \frac{1}{\theta} \mathbf{Q} \nabla \theta - \rho_0 \left( \frac{\partial \mathbf{u}}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0 \]
\[ \frac{1}{2} \Sigma : \frac{\partial \mathbf{C}}{\partial t} - \frac{1}{\theta} \mathbf{Q} \nabla \theta - \rho_0 \left( \frac{\partial \mathbf{u}}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0 \]
\[ S \cdot \frac{\partial \mathbf{F}}{\partial t} - \frac{1}{\theta} \mathbf{Q} \nabla \theta - \rho_0 \left( \frac{\partial \mathbf{u}}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0 \]

Observer changes
\[ \mathbf{F}^* = \mathbf{QF} \]
\[ \mathbf{B}^* = \mathbf{F}^* \mathbf{F}^{*T} = \mathbf{QBQ}^T \]
\[ \mathbf{C}^* = \mathbf{F}^{*T} \mathbf{F}^* = \mathbf{C} \]
\[ \mathbf{\Sigma}^* = \mathbf{\Sigma} \]
General structure of constitutive equations

Assume Local Action; perfect reversibility; no history (rate dep.)

Hence \( \psi = \hat{\psi}(F, \theta) \quad \varepsilon = \hat{\varepsilon}(F, \theta) \)

Frame Indifference

\[ \psi^* = \psi = \hat{\psi}(QF, \theta) = \hat{\psi}(F, \theta) \quad \forall \: Q \in \text{orth}^+ \]

Recall \( F = R \sqrt{c} \quad F = RU \quad U = \sqrt{c} \)

\[ \Rightarrow \hat{\psi}(QR \sqrt{c}, \theta) = \hat{\psi}(F, \theta) \]

This holds for \( Q = R^T \Rightarrow \hat{\psi}(F, \theta) = \hat{\psi}(C, \theta) \)

This applies to all other functions as well.
Dissipation Inequality

\[ \frac{1}{2} \dot{\epsilon} : \frac{dc}{dt} - \frac{1}{\rho} \frac{\partial}{\partial \theta} \cdot \nabla \vartheta \cdot -p_0 \left( \frac{\partial \psi}{\partial t} + \frac{s \partial \vartheta}{\partial t} \right) \geq 0 \]

\[ \frac{\partial \psi}{\partial t} : \frac{\partial \psi}{\partial c} + \frac{\partial \psi}{\partial \vartheta} \frac{\partial \vartheta}{\partial t} \]

\[ \left( \frac{1}{2} \dot{\epsilon} - \frac{p_0 \partial \psi}{\partial c} \right) : \frac{\partial c}{\partial t} - \frac{p_0}{\theta} \left( \frac{\partial \psi}{\partial \theta} + \frac{s}{\theta} \right) \frac{\partial \vartheta}{\partial t} - \frac{1}{\theta} \frac{\partial}{\partial \theta} \cdot \nabla \vartheta \geq 0 \]

\[ \Rightarrow \dot{\epsilon} = 2p_0 \frac{\partial \psi}{\partial c} \quad s = -\frac{\partial \psi}{\partial \vartheta} \]
Other equivalent forms

\[ \frac{1}{\eta} \frac{\partial \mathbf{C}}{\partial t} = \mathbf{S} \cdot \frac{\partial \mathbf{F}}{\partial t} \]

\[ \Rightarrow \mathbf{D} \mathbf{I} \text{ implies } \mathbf{S} = \rho_0 \mathbf{(D} \mathbf{\psi)}^T = \mathbf{S}_{ij} = \rho_0 \frac{\partial \psi}{\partial F_{ji}} \]

Also

\[ \sigma = \frac{1}{J} \mathbf{F} \mathbf{(D} \mathbf{\psi)}^T \]

Natural Reference configuration

Note that \( \hat{\mathbf{e}}(I, \Theta) \neq 0 \) i.e. ref config is not stress free

A ref config with \( \hat{\mathbf{e}}(I, \Theta) = 0 \) is a "natural" ref config
An isotropic material has $\psi$ or $\Sigma$ that are independent of orientation of material wrt principal strain direction.

This requires $\hat{\psi}(FR, \Theta) = \hat{\psi}(F, \Theta)$

or $\hat{\psi}(F^T F, \Theta) = \hat{\psi}(R^T F^T F R, \Theta) = \hat{\psi}(P^T F, \Theta)$

$\Rightarrow \hat{\psi}(R^T C R, \Theta) = \hat{\psi}(C, \Theta)$ holds if $\text{orth}^+ R$

This means $\hat{\psi}$ must be an isotropic function

$\hat{\psi}$ must be a function of the invariants of $C$

Similarly $\Sigma$ can only depend on invariants of $C$
Observations from experiment

For most elastomers:

1. Heat capacity $c_v$ is independent of strain.
2. Internal energy $\varepsilon$ is also independent of strain.

Hence

$$\varepsilon = U + \Theta S$$

$$c_v = \frac{\partial \varepsilon}{\partial \Theta}$$

$$s = -\frac{\partial U}{\partial \Theta}$$

$$c_v = \frac{\partial \varepsilon}{\partial \Theta} + \frac{\partial U}{\partial \Theta} = \Theta \frac{\partial s}{\partial \Theta}$$

Hence

$$s = \int \frac{c_v}{\Theta} d\Theta + g(C)$$

Hence

$$\psi = \Theta g(C) + f(\Theta)$$
Constitutive equations used in practice

Preliminaries: Recall "standard" invariants of $C$ are (same as invariants of $B$)

\begin{align*}
I_1 &= \text{tr}(B) = \text{tr}(C) \\
I_2 &= \frac{1}{2} (I_1^2 - B : B) = \frac{1}{2} (I_1^2 - C : C) \\
I_3 &= \det (B) = \det (C)
\end{align*}

The volumetric and shear measures of strain do not decouple with this set

e.g. let $F = \beta I$ \quad $B = \beta^2 I = C$

\begin{align*}
\Rightarrow I_1 &= 3\beta^2 \\
I_2 &= \frac{1}{2} (9\beta^4 - 3\beta^4) \\
I_3 &= \beta^6
\end{align*}
Another set of invariants (used in ABAQUS) can be used:

$$\bar{I}_1 = \frac{\text{tr}(\mathbf{B})}{J^{2/3}} = \frac{\text{tr}(\mathbf{C})}{J^{4/3}}$$

$$\bar{I}_2 = \frac{1}{2} \left( \bar{I}_1^2 - \frac{\mathbf{B} : \mathbf{B}}{J^{4/3}} \right) = \frac{1}{2} \left( \bar{I}_1^2 - \frac{\mathbf{C} : \mathbf{C}}{J^{4/3}} \right)$$

$$\bar{I}_3 = J$$

For a volumetric deformation $\mathbf{F} = \beta \mathbf{I}$

$$\bar{I}_1 = 3 \quad \bar{I}_2 = 3 \quad , \quad \bar{I}_3 = \beta^3 = J$$

Finally, many material models use eigenvalues of $\mathbf{B}$ and $\mathbf{C}$.
Let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of $U$

Let $\mathbf{b}^{(i)}$, $i = 1, 2, 3$ be principal directions of $B$

Now we can express $p_0(\mathbf{y}) = \Theta p_0 g(\mathbf{y})$. Denote this by $U$

Can use $U(I_1, I_2, I_3) = \mathbf{\bar{U}}(\mathbf{\bar{I}}_1, \mathbf{\bar{I}}_2, \mathbf{\bar{I}}_3) = \hat{U}(\lambda_1, \lambda_2, \lambda_3)$