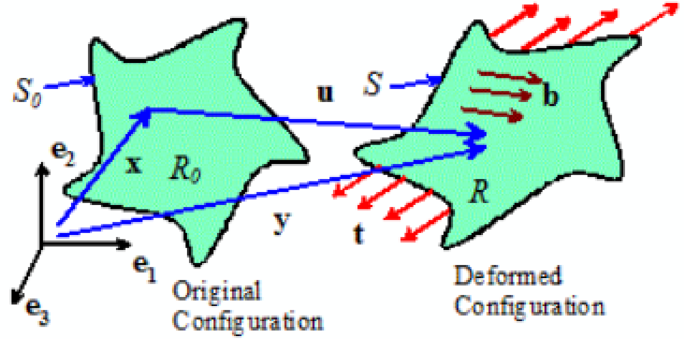
Review: Constitutive models for hyperelastic materials



Standard Invariants:

$$I_1 = \mathrm{trace}(\mathbf{B}) = \mathrm{trace}(\mathbf{C})$$

$$I_2 = \frac{1}{2}\left(I_1^2 - \mathbf{B}:\mathbf{B}\right) = \frac{1}{2}\left(I_1^2 - \mathbf{C}:\mathbf{C}\right)$$

$$I_3 = \det\mathbf{B} = \det\mathbf{C}$$

Modified Invariants:

$$\bar{I}_1 = \frac{I_1}{J^{2/3}} = \frac{tr(\mathbf{B})}{J^{2/3}} = \frac{tr(\mathbf{C})}{J^{2/3}}$$

$$\bar{I}_2 = \frac{I_2}{J^{4/3}} = \frac{1}{2}\left(\bar{I}_1^2 - \frac{\mathbf{B}:\mathbf{B}}{J^{4/3}}\right) = \frac{1}{2}\left(\bar{I}_1^2 - \frac{\mathbf{C}:\mathbf{C}}{J^{4/3}}\right)$$

$$J = \det\mathbf{F} = \sqrt{\det\mathbf{B}} = \sqrt{\det\mathbf{C}}$$

Spectral decomposition of **B**: $\mathbf{B} = \lambda_1^2 \mathbf{b}^{(1)} \otimes \mathbf{b}^{(1)} + \lambda_2^2 \mathbf{b}^{(2)} \otimes \mathbf{b}^{(2)} + \lambda_3^2 \mathbf{b}^{(3)} \otimes \mathbf{b}^{(3)}$

Helmholtz free energy: $\rho_0 \psi(\mathbf{C},\theta) = \rho_0\left(\theta g(\mathbf{C}) + f(\theta)\right) = U(I_1, I_2, I_3) = \bar{U}(\bar{I}_1, \bar{I}_2, J) = \tilde{U}(\lambda_1, \lambda_2, \lambda_3)$

Cauchy stress: $\sigma_{ij} = \frac{1}{J} F_{ik} \frac{\partial U}{\partial F_{jk}}$

Explicit formulas for Cauchy stress

Useful Derivatives

$$\frac{\partial I_1}{\partial F_{ij}} = 2F_{ij}, \; \frac{\partial I_2}{\partial F_{ij}} = 2\left(I_1 F_{ij} - B_{ik}F_{kj}\right), \; \frac{\partial I_3}{\partial F_{ij}} = 2I_3 F_{ji}^{-1}$$

$$\frac{\partial J}{\partial F_{ij}} = J F_{ji}^{-1} \qquad \frac{\partial \bar{I}_1}{\partial F_{ij}} = \frac{1}{J^{2/3}}\frac{\partial I_1}{\partial F_{ij}} - \frac{2I_1}{3J^{5/3}}\frac{\partial J}{\partial F_{ij}} = \frac{2}{J^{2/3}}\left(F_{ij} - \frac{I_1}{3}F_{ji}^{-1}\right) = \frac{2}{J^{2/3}}F_{ij} - \frac{2}{3}\bar{I}_1 F_{ji}^{-1}$$

$$\frac{\partial \bar{I}_2}{\partial F_{ij}} = \frac{1}{J^{4/3}}\frac{\partial I_2}{\partial F_{ij}} - \frac{4I_2}{3J^{7/3}}\frac{\partial J}{\partial F_{ij}} = \frac{2}{J^{4/3}}\left(I_1 F_{ij} - B_{ik}F_{kj} - \frac{2I_2}{3}F_{ji}^{-1}\right) = \frac{2}{J^{2/3}}\bar{I}_1 F_{ij} - \frac{2}{J^{4/3}}B_{ik}F_{kj} - \frac{4\bar{I}_2}{3}F_{ji}^{-1}$$

Cauchy Stress formulas

For $U(I_1, I_2, I_3)$
$$\sigma_{ij} = \frac{1}{\sqrt{I_3}}F_{ik}\frac{\partial U}{\partial F_{jk}} = \frac{2}{\sqrt{I_3}}\left[\left(\frac{\partial U}{\partial I_1} + I_1\frac{\partial U}{\partial I_2}\right)B_{ij} - \frac{\partial U}{\partial I_2}B_{ik}B_{kj}\right] + 2\sqrt{I_3}\frac{\partial U}{\partial I_3}\delta_{ij}$$

For $\bar{U}(\bar{I}_1, \bar{I}_2, J)$
$$\sigma_{ij} = \frac{1}{J}F_{ik}\frac{\partial \bar{U}}{\partial F_{jk}} = \frac{2}{J^{5/3}}\left(\frac{\partial \bar{U}}{\partial \bar{I}_1} + \bar{I}_1\frac{\partial \bar{U}}{\partial \bar{I}_2}\right)B_{ij} - \frac{2}{3J}\left(\bar{I}_1\frac{\partial \bar{U}}{\partial \bar{I}_1} + 2\bar{I}_2\frac{\partial \bar{U}}{\partial \bar{I}_2}\right)\delta_{ij} - \frac{2}{J^{7/3}}\frac{\partial \bar{U}}{\partial \bar{I}_2}B_{ik}B_{kj} + \frac{\partial \bar{U}}{\partial J}\delta_{ij}$$

For $\tilde{U}(\lambda_1, \lambda_2, \lambda_3)$
$$\sigma_{ij} = \frac{\lambda_1}{\lambda_1\lambda_2\lambda_3}\frac{\partial \tilde{U}}{\partial \lambda_1}b_i^{(1)}b_j^{(1)} + \frac{\lambda_2}{\lambda_1\lambda_2\lambda_3}\frac{\partial \tilde{U}}{\partial \lambda_2}b_i^{(2)}b_j^{(2)} + \frac{\lambda_3}{\lambda_1\lambda_2\lambda_3}\frac{\partial \tilde{U}}{\partial \lambda_3}b_i^{(3)}b_j^{(3)}$$

Derivation of σ for $U(I_1, I_2, I_3)$

* General formula $\sigma_{ij} = \frac{1}{J} F_{ik} \frac{\partial U}{\partial F_{jk}}$

$$\sigma_{ij} = \frac{2}{J} F_{ik} \left(\frac{\partial U}{\partial I_1} F_{jk} + \frac{\partial U}{\partial I_2} \left(I_1 F_{jk} - B_{jn} F_{nk} \right) + \frac{\partial U}{\partial I_3} 2 I_3 F^{-1}_{kj} \right)$$

$$= \frac{2}{\sqrt{I_3}} \left(\frac{\partial U}{\partial I_1} B_{ij} + \frac{\partial U}{\partial I_2} \left(I_1 B_{ij} - B_{jn} B_{in} \right) + 2 I_3 \frac{\partial U}{\partial I_3} \delta_{ij} \right)$$

Examples of practical hyperelastic potentials

Neo-Hookean (generalized)

$$\bar{U} = \frac{\mu_1}{2} (\bar{I}_1 - 3) + \frac{K_1}{2} (J-1)^2$$

μ_1 = shear modulus
k_1 = Bulk modulus } small deformations

Use only with $k_1 >> \mu_1$

Mooney - Rivlin :

$$\bar{U} = \frac{\mu_1}{2}(\bar{I}_1 - 3) + \frac{\mu_2}{2}(\bar{I}_2 - 3) + \frac{k_2}{2}(J-1)^2$$

Here $\mu_1 + \mu_2$ = shear modulus

Polynomial potential

$$\bar{U} = \sum_{i+j=1}^{N} C_{ij} (\bar{I}_1 - 3)^i (\bar{I}_2 - 3)^j + \sum_{i=1}^{M} k_i (J-1)^{2i}$$

Ogden : $\hat{U} = \sum_{i=1}^{N} \frac{2\mu_i}{\alpha_i^2}\left(\left(\frac{\lambda_1}{J}\right)^{\alpha_i} + \left(\frac{\lambda_2}{J}\right)^{\alpha_i} + \left(\frac{\lambda_3}{J}\right)^{\alpha_i} - 3\right)$

$$+ \sum_{i=1}^{N} K_i (J-1)^{2i}$$

α_i, μ_i - constants - fit to data

Incompressible materials (for analytical solutions)

(1) $J = 1$ (constraint)

(2) pressure (hydrostatic stress) can't be determined from strains (Lagrange multiplier)

Determine p from equilibrium, boundary conditions

For incompressible materials we can still use usual formulas but replace hydrostatic terms with unknown p

eg for $U(I_1, I_2, I_3=1)$

$$\sigma_{ij} = 2\left(\frac{\partial U}{\partial I_1} B_{ij} + \frac{\partial U}{\partial I_2}\left(B_{ij} - B_{in}B_{nj}\right)\right) + p\delta_{ij}$$

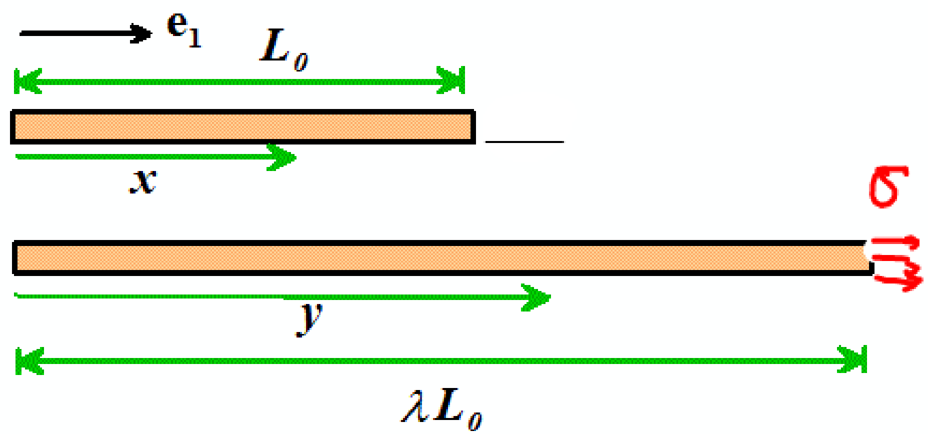
Applications to simple deformations

Assume incompressible

Uniaxial Tension

Define $\underset{\sim}{P} = \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1$

$\underset{\sim}{Q} = (\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3)$



$$F = \lambda P + \frac{1}{\sqrt{\lambda}} Q$$

$$B = \lambda^2 P + \frac{1}{\lambda} Q$$

$$I_1 = \lambda^2 + \frac{2}{\lambda}$$

Neo-Hookean material $\quad \sigma = \mu_1 B + p I$

Find p from $\sigma_{22} = \sigma_{33} = 0 \quad \Rightarrow \frac{\mu_1}{\lambda} + p = 0$

$$p = -\frac{\mu_1}{\lambda}$$

$$\boxed{\begin{array}{l} \sigma = \mu_1\left(\lambda^2 - \frac{1}{\lambda}\right) \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 \\ \text{Also } \; S = J F^{-1} \sigma = \mu_1 \left(\lambda - \frac{1}{\lambda^2}\right) \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 \end{array}}$$

Mooney - Rivlin

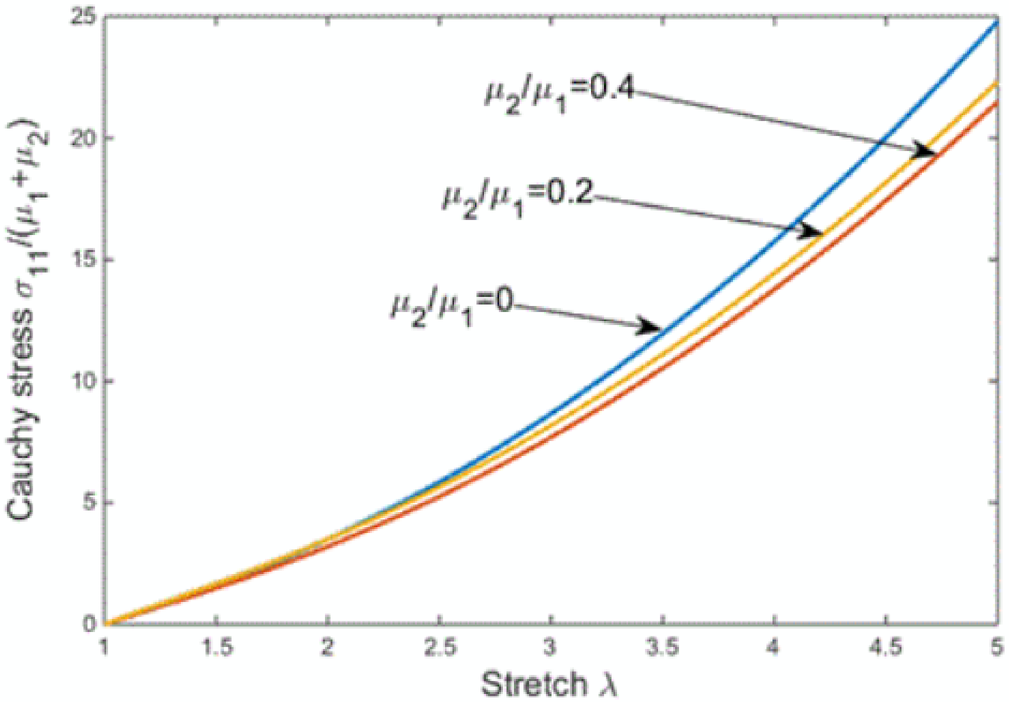
$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}}\left(B_{ij} - \frac{1}{3}B_{kk}\delta_{ij}\right) + \frac{\mu_2}{J^{7/3}}\left(B_{kk}B_{ij} - \frac{1}{3}[B_{kk}]^2\delta_{ij} - B_{ik}B_{kj} + \frac{1}{3}B_{kn}B_{nk}\delta_{ij}\right) + p\delta_{ij}$$

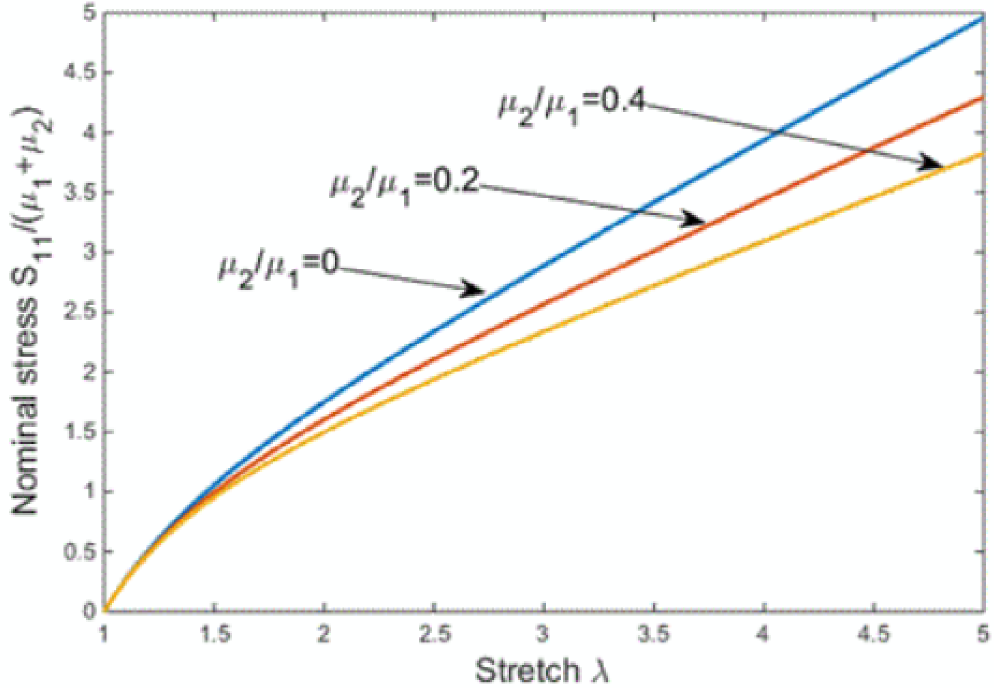
Incompressible $\Rightarrow \sigma = \mu_1 B + \mu_2\left(\left(\lambda^2 + \frac{2}{\lambda}\right)B - B^2\right) + pI$

As before p found from $\sigma_{22} = \sigma_{33} = 0$

$$p = -\frac{\mu_1}{\lambda} - \mu_2\left(\left(\lambda^2 + \frac{2}{\lambda}\right)\frac{1}{\lambda} - \frac{1}{\lambda^2}\right)$$

$$\boxed{\begin{aligned}\sigma &= \left[\mu_1\left(\lambda^2 - \frac{1}{\lambda}\right) + \mu_2\left(\lambda - \frac{1}{\lambda^2}\right)\right]\underline{e}_1 \otimes \underline{e}_1 \\ S &= \left[\mu_1\left(\lambda - \frac{1}{\lambda^3}\right) + \mu_2\left(1 - \frac{1}{\lambda^3}\right)\right]\underline{e}_1 \otimes \underline{e}_1\end{aligned}}$$

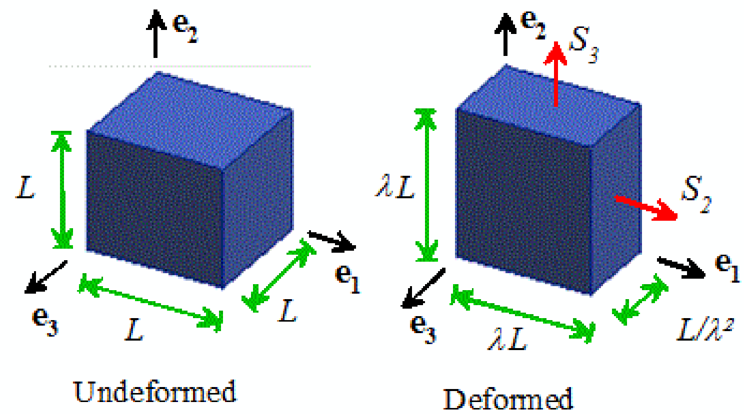




Biaxial stretching

$$P = (\underline{e}_1 \otimes \underline{e}_1 + \underline{e}_2 \otimes \underline{e}_2)$$

$$Q = \underline{e}_3 \otimes \underline{e}_3$$



$$F = \lambda P + \frac{1}{\lambda^2} Q \qquad B = \lambda^2 P + \frac{1}{\lambda^4} Q$$

$$\boxed{\begin{aligned} \sigma &= \left[\mu_1 \left(\lambda^2 - \frac{1}{\lambda^4}\right) + \mu_2 \left(\lambda^4 - \frac{1}{\lambda^2}\right)\right] P \\ S &= \left[\mu_1 \left(\lambda - \frac{1}{\lambda^5}\right) + \mu_2 \left(\lambda^3 - \frac{1}{\lambda^3}\right)\right] P \end{aligned}}$$