Review – Vector Calculus

Scalar Field: $\phi(x_1, x_2, x_3)$

Gradient: $[\nabla \phi]_i = \dfrac{\partial \phi}{\partial x_i}$

Vector Field: $\mathbf{v}(x_1, x_2, x_3)$

Gradient: $[\nabla \mathbf{v}]_{ij} = \dfrac{\partial v_i}{\partial x_j}$ $\quad \nabla \mathbf{v} \cdot \mathbf{n} = \underset{\in \to 0}{Lim} \dfrac{\mathbf{v}(\mathbf{x} + \in \mathbf{n}) - \mathbf{v}(\mathbf{x})}{\in}$ $\quad \forall\ \mathbf{n}:\ \mathbf{n} \cdot \mathbf{n} = 1$

Divergence: $\nabla \cdot \mathbf{v} = trace(\nabla \mathbf{v}) \equiv \dfrac{\partial v_i}{\partial x_i}$

Curl: $[\nabla \times \mathbf{v}]_i = \in_{ijk} \dfrac{\partial v_k}{\partial x_j}$

* Laplacian

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x_i \partial x_i} \equiv \nabla \cdot \nabla \phi$$

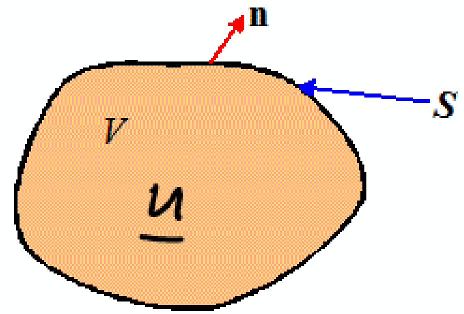
$$\nabla^2 \underline{u} \equiv \frac{\partial^2 u_i}{\partial x_j \partial x_j} \not\equiv \nabla \cdot \nabla \underline{u}$$

* Comma notation for derivatives

$$\phi_{,i} = \frac{\partial \phi}{\partial x_i} \qquad u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

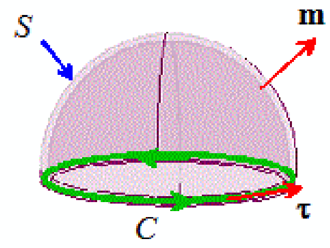
* Divergence Theorem

$$\int_V \nabla \cdot \underline{v} \, dV = \int_S \underline{v} \cdot \underline{n} \, dA$$



* Stokes Theorem

$$\int_S (\nabla \times \underline{u}) \cdot \underline{m} \, dA = \oint_C \underline{u} \cdot \underline{\tau} \, ds$$



Index Notation Examples

Example 1

1. Which of the following equations are valid expressions using index notation? If you decide an expression is invalid, state which rule is violated.

(a) $\sigma_{ij} = C_{klij}\varepsilon_{kl}$ ✓ (b) $\cancel{\epsilon_{kkk}} = 0$ (c) $\dfrac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \dfrac{\partial^2 u_i}{\partial t^2}$ ✓ (d) $\epsilon_{ijk}\epsilon_{ijk} = 6$ ✓

can't repeat an index 3x

Example 2

Let $R = \sqrt{x_k x_k}$. Calculate $\dfrac{\partial R}{\partial x_i}$ and $\dfrac{\partial^2 R}{\partial x_i \partial x_j}$.

Note $\dfrac{\partial x_k}{\partial x_i} = \delta_{ik}$

$$\frac{\partial R}{\partial x_i} = \frac{1}{2}\frac{1}{\sqrt{x_n x_n}}\left(\delta_{ik} x_k + x_k \delta_{ik}\right)$$

$$= \frac{x_i}{R}$$

$$\frac{\partial^2 R}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{R} - \frac{x_i}{R^2}\frac{x_j}{R} = \frac{\delta_{ij}}{R} - \frac{x_i x_j}{R^3}$$

Useful Formulas

$$\in_{ijk} \in_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk}$$

$$\in_{ijk} \in_{lmn} = \delta_{il}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right) + \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)$$

Example 3 Show that $\nabla \times (\mathbf{u} \times \mathbf{v}) = (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v} + (\nabla \mathbf{u})\mathbf{v} - (\nabla \mathbf{v})\mathbf{u}$

$$\epsilon_{ijk} \frac{\partial}{\partial x_j}\left[\epsilon_{klm}(u_l v_m)\right] = \epsilon_{kij}\,\epsilon_{klm} \frac{\partial}{\partial x_j}(u_l v_m)$$

$$= \delta_{il}\delta_{jm} \frac{\partial}{\partial x_j} u_l v_m - \delta_{im}\delta_{jl} \frac{\partial}{\partial x_j}(u_l v_m)$$

$$= \frac{\partial (u_i v_j)}{\partial x_j} - \frac{\partial (u_j v_i)}{\partial x_j}$$

$$\frac{\partial v_j}{\partial x_j} u_i - \frac{\partial u_j}{\partial x_j} v_i + \frac{\partial u_i}{\partial x_j} v_j - \frac{\partial v_i}{\partial x_j} u_j$$

2.2 Tensors

Definition: Linear vector valued function defined on the set of all vectors

Denoted $\underline{u} = S(\underline{v}) \equiv \underline{u} = S\underline{v} = S \cdot \underline{v}$

* Linearity $\Rightarrow$ $S(\underline{u} + \underline{v}) = S(\underline{u}) + S(\underline{v})$

$S(\alpha \underline{u}) = \alpha S(\underline{u})$

* Example: Gradient of a vector

$$d\underline{u} = (\nabla \underline{u})\, d\underline{x}$$

Tensor components (othonormal basis)

Let $\{\underset{\sim}{m}_1, \underset{\sim}{m}_2, \underset{\sim}{m}_3\}$ be a basis

$$\underset{\sim}{m}_i \cdot \underset{\sim}{m}_j = \delta_{ij}$$

$$\underset{\sim}{u} = S\,\underset{\sim}{v} = u_i\,\underset{\sim}{m}_i = S\, v_i\,\underset{\sim}{m}_i$$

$$\underset{\sim}{m}_i \cdot \underset{\sim}{m}_k\, u_i = (\underset{\sim}{m}_k \cdot S\,\underset{\sim}{m}_i)\, v_i$$

$$u_k = \underbrace{(\underset{\sim}{m}_k \cdot S\,\underset{\sim}{m}_i)\, v_i}_{3\times 3 \text{ matrix}}$$

Index Notation

$$u_k = S_{ki}\, v_i$$

$$(S_{ki} = \underset{\sim}{m}_k \cdot S\,\underset{\sim}{m}_i)$$

Dyadic Product (Tensor Product) of two vectors

Let $\underline{a}, \underline{b}$ be two vectors

Then $\underline{\underline{S}} = \underline{a} \otimes \underline{b}$: $\underline{\underline{S}}\,\underline{u} = \underline{a}\,(\underline{b}\cdot\underline{u}) \quad \forall\, \underline{u}$

$$S_{ij} = a_i b_j$$

Not all tensors can be created from two vectors since

$$\underline{\underline{S}}\,(\underline{b}\times\underline{v}) = \underline{a}\,(\underline{b}\cdot\underline{b}\times\underline{v}) = 0$$

However, we can express all tensors as linear combinations of products of basis vectors

$\{\underline{m}_1, \underline{m}_2, \underline{m}_3\}$ — basis (not orthonormal)

$$S = S^{ij}\, \underline{m}_i \otimes \underline{m}_j$$

$S_{kl} = \underline{m}_k \cdot S\, \underline{m}_l$
$= \underline{m}_k \cdot \underline{m}_i S^{ij}\, \underline{m}_j \cdot \underline{m}_l$

For an orthonormal basis

$$S = S_{ij}\, \underline{m}_i \otimes \underline{m}_j$$

Basis Change Formula for tensors

Let $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ $\{\underline{m}_1, \underline{m}_2, \underline{m}_3\}$
be bases with $\underline{e}_i \cdot \underline{e}_j = \underline{m}_i \cdot \underline{m}_j = \delta_{ij}$

Components $\underset{=}{S}_{ij}^{m}$; $\underset{=}{S}_{ij}^{e}$ in two bases

$$S = S_{ij}^{m}\, \underline{m}_i \otimes \underline{m}_j = S_{ij}^{e}\, \underline{e}_i \otimes \underline{e}_j$$

Hence

$$\underbrace{(\underline{m}_i \cdot \underline{m}_k)}_{\delta_{ik}} \underbrace{(\underline{m}_j \cdot \underline{m}_\ell)}_{\delta_{j\ell}} S_{ij}^{\underline{m}} = \underbrace{\underline{m}_k \cdot \underline{e}_i}_{Q_{ki}} \; S_{ij}^{\underline{e}} \; \underbrace{\underline{e}_j \cdot \underline{m}_\ell}_{Q_{\ell j}}$$

$$\boxed{\begin{aligned} S_{k\ell}^{\underline{m}} &= Q_{ki} \, S_{ij}^{\underline{e}} \, Q_{\ell j} \\ [S^{\underline{m}}] &= [Q] \, [S^{\underline{e}}] \, [Q]^T \end{aligned}}$$

Basic Tensor Operations

Let $\mathbf{U}, \mathbf{S}, \mathbf{T}$ be tensors with components U_{ij}, S_{ij}, T_{ij}
Let $\mathbf{u}, \mathbf{v}$ be vectors with components v_i, u_i
in an orthonormal basis

Addition: $\mathbf{U} = \mathbf{S} + \mathbf{T}$ $\quad U_{ij} = S_{ij} + T_{ij}$

Tensor-vector products: $\mathbf{v} = \mathbf{S}\mathbf{u}$ $\quad v_i = S_{ij} u_j$ $\quad \mathbf{v} = \mathbf{u}\mathbf{S}$ $\quad v_i = u_j S_{ji}$

Product of two tensors: $\mathbf{U} = \mathbf{T}\mathbf{S}$ $\quad U_{ij} = T_{ik} S_{kj}$ $\quad$ NB: $(\mathbf{TS} \neq \mathbf{ST})$

Transpose: $\mathbf{S}^T$ $\quad \mathbf{u} \cdot \mathbf{S}^T = \mathbf{S} \cdot \mathbf{u}$ $\quad S_{ij}^T = S_{ji}$

Note: $(\mathbf{ST})^T = \mathbf{T}^T \mathbf{S}^T$ (show with index notation)

Trace: $\mathrm{trace}(\mathbf{S}) = S_{11} + S_{22} + S_{33}$ $\quad \mathrm{trace}(\mathbf{S}) = S_{kk}$