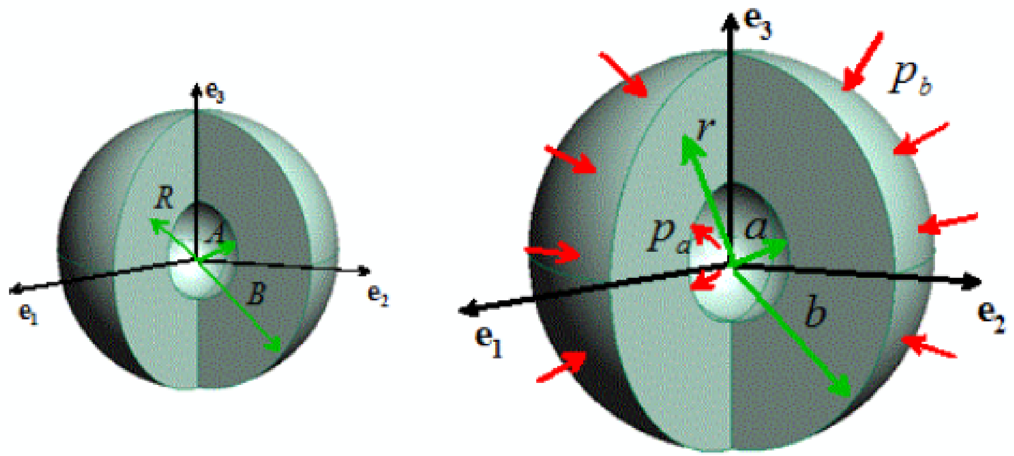
Examples of solutions for hyperelastic solids

Example 1: Pressurized Shell



* Assumptions:

(1) Incompressible

(2) Neo-Hookean solid

$$\underline{x} = R\,\underline{e_R} \qquad \underline{y} = r(R)\,\underline{e_R}$$

* Incompressibility: $\frac{4}{3}\pi(r^3 - a^3) = \frac{4}{3}\pi(R^3 - A^3)$

$$r = (R^3 + a^3 - A^3)^{1/3}$$

$$R = (r^3 + A^3 - a^3)^{1/3}$$

$$F = \nabla y = r\,\underline{e_R} \otimes \left(\frac{\partial}{\partial R}\underline{e_R} + \frac{1}{R}\frac{\partial}{\partial \theta}\underline{e_\theta} + \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}\underline{e_\phi} \right)$$

$$F = \frac{R^2}{r^2} \underline{e}_R \otimes \underline{e}_R + \frac{r}{R}\left(\underline{e}_\theta \otimes \underline{e}_\theta + \underline{e}_\phi \otimes \underline{e}_\phi\right)$$

$$B = FF^T = \frac{R^4}{r^4} \underline{e}_R \otimes \underline{e}_R + \frac{r^2}{R^2}\left(\underline{e}_\theta \otimes \underline{e}_\theta + \underline{e}_\phi \otimes \underline{e}_\phi\right)$$

* Stress-strain : $\sigma = \mu B + pI$ $\quad$ (0)

* Equilibrium: $\frac{\partial \sigma_{rr}}{\partial r} + \frac{2}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 0$

$$\frac{\partial \sigma_{rr}}{\partial r} + 2\mu\left(\frac{R^4}{r^5} - \frac{r}{R^2}\right) = 0$$

Recall R depends on r

$$\sigma_{rr} = \mu\left(\frac{2R}{r} + \frac{R^4}{2r^4}\right) + C$$

* Boundary conditions:

$$\mu\left(2\frac{A}{a} + \frac{A^4}{2a^4}\right) + C = -p_a$$

$$\mu\left(\frac{2B}{b} + \frac{B^4}{2a^4}\right) + C = -p_b$$

Let $a/A = \alpha \qquad b/B = \beta$

$$\left(\frac{2}{\beta} - \frac{2}{\alpha}\right) + \left(\frac{1}{2\beta} - \frac{1}{2\alpha}\right) = p_a - p_b \qquad (1)$$

Also $\quad b^3 - a^3 = B^3 - A^3$

$$B^3\beta^3 - A^3\alpha^3 = (B^3 - A^3)$$

$$\frac{B}{A}\beta^3 - \alpha^3 = \left(\frac{B}{A}\right)^3 - 1 \qquad (2)$$

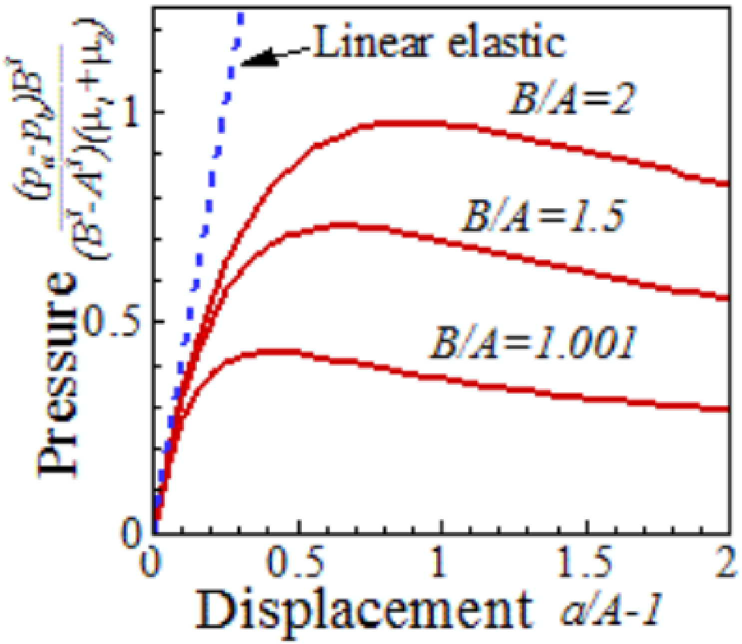
Two equations for α, β ; can solve

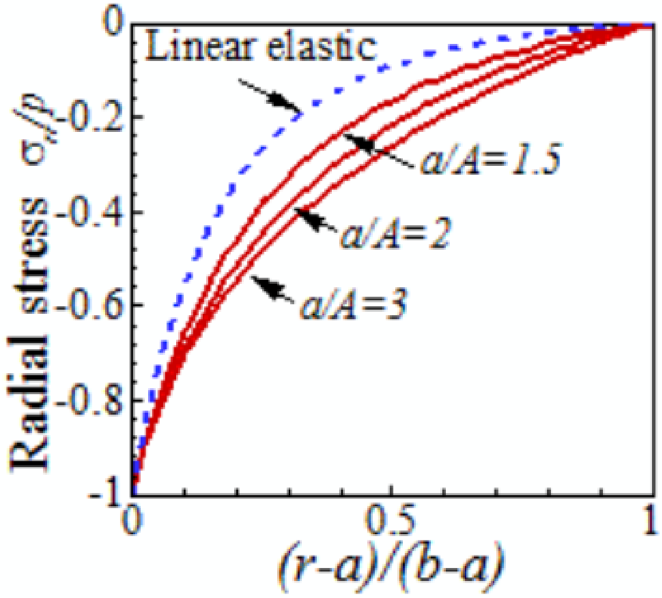
To plot:

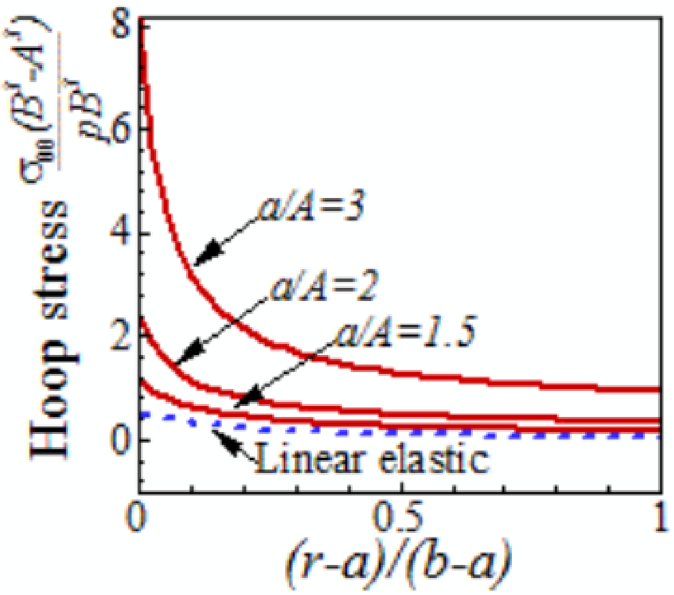
1. Pick α
2. Solve 2 for β
3. 1 gives pressure

Softening caused by change in wall thickness

Stresses follow by solving for C ; subst back into 0



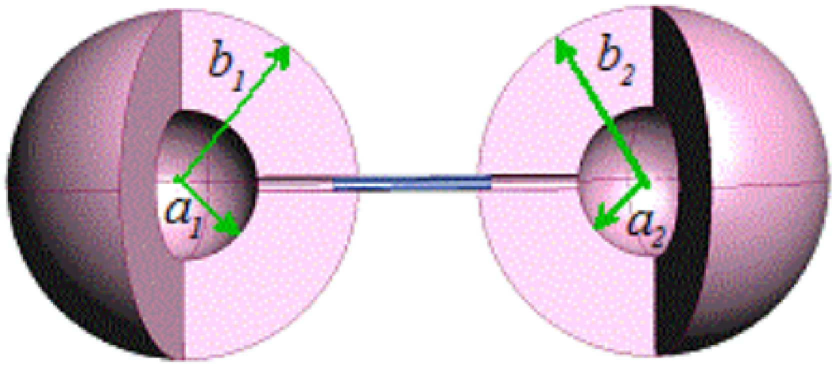




Example 2:

Two spheres ; B, A

Connect with tube



Fill with fluid volume $V = 2\omega^3 \frac{4}{3}\pi A^3$

$\omega > 1$

Find a_1, a_2

For simplicity assume thin walled sphere

Volume $\frac{4}{3}\pi a_1^3 + \frac{4}{3}\pi a_2^3 = 2\omega^3 \frac{4}{3}\pi A^3$

$\alpha_1^3 + \alpha_2^3 = 2 \qquad \alpha = \frac{a}{\omega A}$

Find α_1, α_2 from energy

① Strain energy density $U = \frac{\mu}{2}(I_1 - 3)$

$U \approx \frac{\mu}{2}\left(\frac{A^4}{a^4} + 2\frac{a^2}{A^2}\right)$ (can forget the 3)

Total strain energy

$$\bar{\Phi} = 4\pi A^2 (B-A)\,\frac{\mu}{2}\left(\frac{1}{\alpha_1^4\omega^4} + \frac{1}{\alpha_2^4\omega^4} + 2(\alpha_1^2+\alpha_2^2)\omega^2\right)$$

$$\frac{1}{2\pi}\,\frac{\bar{\Phi}}{\mu A^2(B-A)} = \frac{1}{\omega^4}\left(\frac{1}{\alpha_1^4} + \frac{1}{(2-\alpha_1^3)^{4/3}} + 2\omega^2\left(\alpha_1^2 + (2-\alpha_1^3)^{2/3}\right)\right)$$

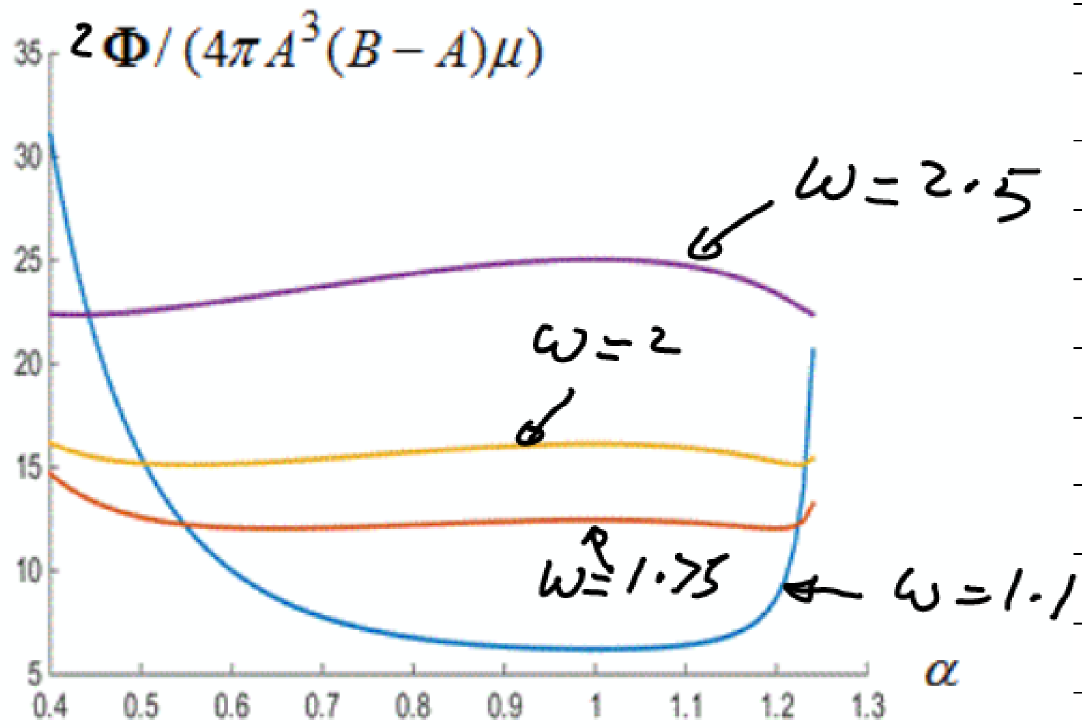
Note :

$\omega < \omega_{crit}$ $\alpha_1 = \alpha_2 = 1$ is a stable minimum

$\omega > \omega_{crit}$

$\alpha_1 = \alpha_2$ unstable equilibrium

Two new stable equilibria



We can find ω_{crit} from $\left.\frac{d^2\bar{\Phi}}{d\alpha_1^2}\right|_{\alpha_1=1} = 0$

$$\omega_{crit} = 7^{1/6} \approx 1.28$$

Can solve $\frac{d\bar{\Phi}}{d\alpha_1} = 0$ to find α_1 given ω

Bifurcation diagram