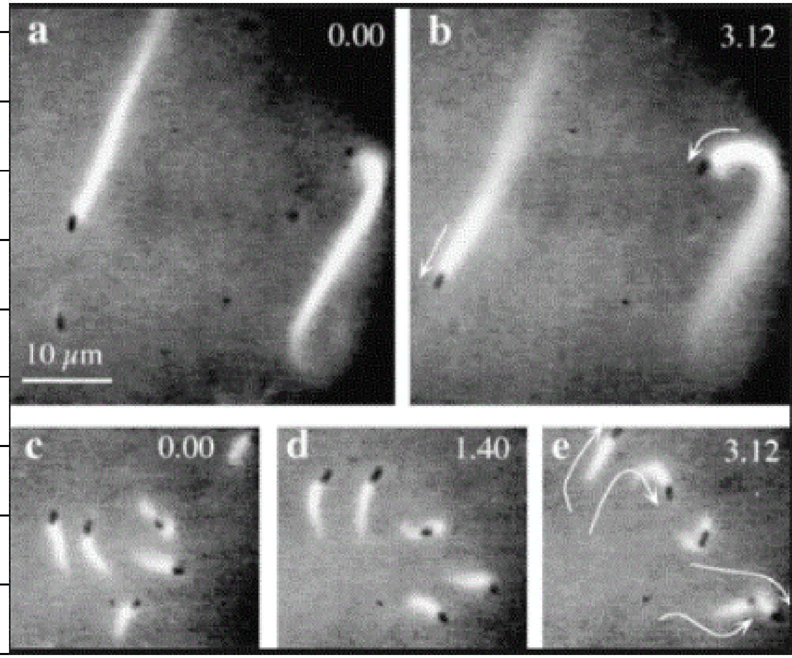
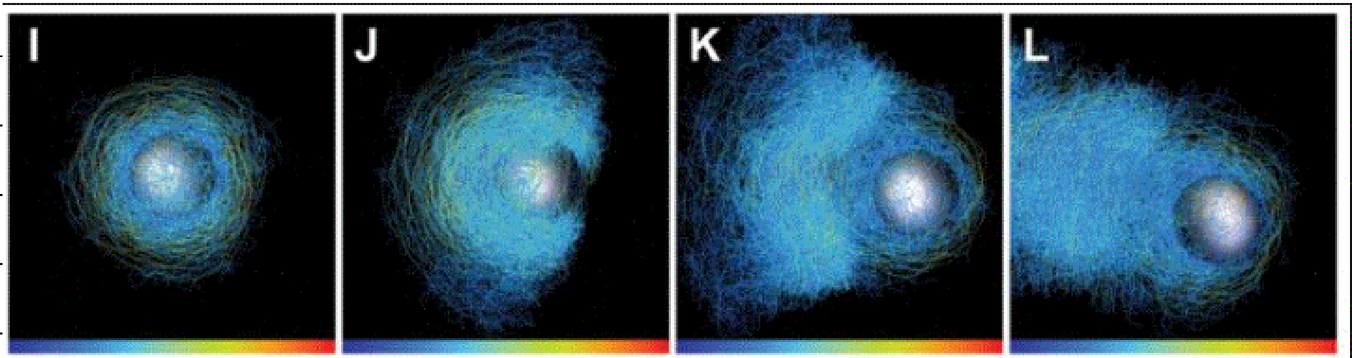
Examples of solutions to boundary value problems for hyperelastic materials

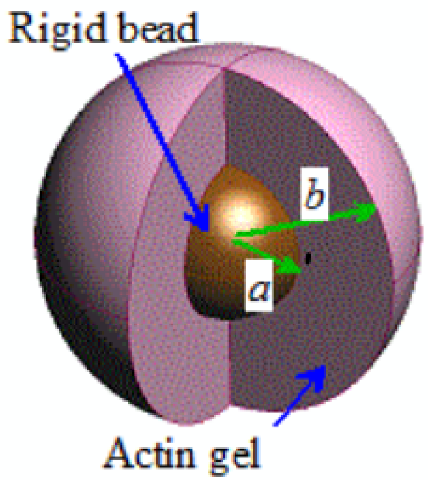
Listeria bacterium is propelled by "comet tail" that consists of a polymer gel forming around the bacterium





- duplicate this process by coating beads with an enzyme
  - spherical gel forms around sphere; eventually breaks symmetry & forms tail.

Goal: Calculate stress in actin gel growing on bead



Assumptions: (1) All gel grows on bead surface
(2) Assume stress is hydrostatic @ bead surface
(3) Assume gel is an incompressible hyperelastic material

① Kinematics: No ref config!

Note that a shell @ radius r started @ radius a

Hence $\lambda_{\theta\theta} = \lambda_{\phi\phi} = \frac{r}{a}$

λ_{rr} from incompressibility $\det(F) \circ \lambda_{rr} = \frac{a^2}{r^2}$

$$\mathbf{B} = FF^T = \frac{a^4}{r^4}\, \underline{e_r} \otimes \underline{e_r} + \frac{r^2}{a^2}\left(\underline{e_\phi} \otimes \underline{e_\phi} + \underline{e_\theta} \otimes \underline{e_\theta} \right)$$

② $\sigma = \mu B + p I$

③ Equilibrium $\quad \dfrac{\partial \sigma_{rr}}{\partial r} + \dfrac{2}{r}\left(\sigma_{rr} - \sigma_{\theta\theta} \right) = 0$

$$\frac{\partial \sigma_{rr}}{\partial r} + 2\mu\left(\frac{a^4}{r^5} - \frac{r}{a^2} \right) = 0$$

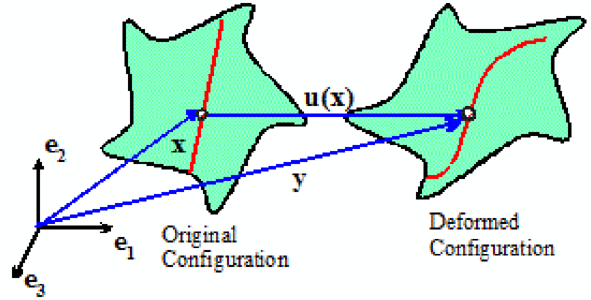
Boundary condition : $\sigma_{rr} = 0$ @ $r = b$

$$\sigma_{rr} = \frac{\mu}{2}\left(\frac{a^4}{r^4} - \frac{a^4}{b^4} \right) + \mu \frac{\left(b^2 - r^2 \right)}{a^2}$$

$$\sigma_{\theta\theta} = \sigma_{rr} + 2\mu\left(\frac{a^4}{r^4} - \frac{r^2}{a^2} \right)$$

Linearized equations of elasticity

Goal: Derive field equations governing small displacements of elastic solids



Assumptions: (1) Solid is stress free @ some reference temp θ_0

(2) Assume $|\nabla \underline{u}| << 1$

Kinematics:

$$F \equiv I + \nabla \underline{u}$$

$$J = \det(F) \approx 1 + \mathrm{tr}(\nabla \underline{u})$$

$$\rho = \rho_0$$

$$C = F^T F \approx I + \nabla \underline{u} + (\nabla \underline{u})^T$$

$$I + 2E$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Stress measures : $\sigma \approx S \approx \Sigma$ (from earlier)

- all stress measures equal σ

BAM : $\sigma = \sigma^T$

BLM : $\underbrace{\nabla \cdot \sigma} + \rho_0 \underline{b} \approx \rho_0 \left.\frac{\partial^2 \underline{u}}{\partial t^2}\right|_{\underline{x}}$

(Replaces $\nabla_y \cdot \sigma$)

Constitutive equations

Helmholtz free energy $\psi = \hat{\psi}(C, \theta)$

$= \hat{\psi}(I + 2\varepsilon, \theta)$

Introduce strain energy density

$\rho_0 \hat{\psi} = U(\underset{\sim}{\varepsilon}, \theta)$

Stress response function $\sigma_{ij} = \hat{\sigma}_{ij}(\varepsilon, \theta)$

Recall $\hat{\Sigma}_{ij} = 2\rho_0 \dfrac{\partial \hat{\psi}}{\partial C_{ij}}$

$$\Rightarrow \quad \hat{\sigma}_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$$

Finally expand $\hat{\sigma}_{ij}$ as Taylor series in ε, θ

Assume $\hat{\sigma}_{ij} = 0$ @ $\varepsilon_{ij} = 0$, $\theta = \theta_0$

$$\hat{\sigma}_{ij} = + \underbrace{\left.\frac{\partial^2 U}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}}\right|_{\substack{\varepsilon = 0 \\ \theta = \theta_0}}}_{C_{ijkl}\text{ - Isothermal moduli}} \varepsilon_{kl} + \underbrace{\left.\frac{\partial^2 U}{\partial \varepsilon_{ij}\, \partial \theta}\right|_{\substack{\varepsilon = 0 \\ \theta = \theta_0}}}_{b_{ij}} (\theta - \theta_0)$$

Hence $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} + b_{ij}(\theta - \theta_0)$

Alternatively define $\alpha_{ij} = -C^{-1}_{ijkl}\,b_{kl}$

(or $C_{ijkl}\,\alpha_{kl} = b_{kl}$)

α_{ij} = thermal expansion coeft

$\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \alpha_{ij}(\theta - \theta_0)\right)$

Note that $C_{ijkl} = C_{klij}$ $\quad C_{ijkl} = C_{jikl}$ $\quad C_{ijkl} = C_{ijlk}$

21 + 6 material properties for a general material

Isotropic Materials C_{ijkl} must be an isotropic tensor

Representation Theorem: $C_{ijkl} = \mu\delta_{il}\delta_{jk} + \gamma\delta_{ik}\delta_{jl} + \lambda\delta_{ij}\delta_{kl}$

Symmetry of $C \Rightarrow$

$$C_{ijkl} = \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + \lambda\delta_{ij}\delta_{kl}$$

(μ, λ) Lame moduli

μ : shear modulus

$$S_{ijkl} = C_{ijkl}^{-1} = \frac{1}{4\mu}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) - \frac{\lambda(\mu+\lambda)}{2\mu(3\mu+2\lambda)}\delta_{ij}\delta_{kl}$$

Other moduli : $E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ - Young's modulus

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \quad \text{- Poissons ratio}$$

$$K = \frac{3\lambda+2\mu}{3} \quad \text{- Bulk modulus.}$$

Summary of field equations

Strain - displacement $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Constitutive eq : $\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \alpha_{kl}(\theta - \theta_0)\right)$

BLM : $\frac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 b_j = \rho_0 \frac{\partial^2 u_j}{\partial t^2}$

Linear PDEs !