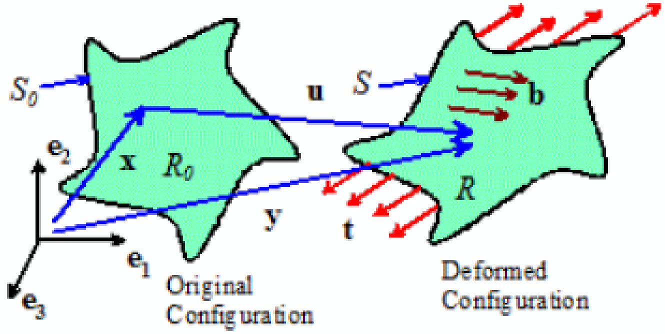
Review: Field equations of linear elasticity



Strain-displacement relation: $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$

Constitutive equation: $\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \alpha_{kl}\Delta\theta)$

Isotropic material: $C_{ijkl} = \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + \lambda\delta_{ij}\delta_{kl}$

$$\nu = \frac{\lambda}{2(\mu+\lambda)} \qquad E = 2\mu(1+\nu)$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta\theta\delta_{ij}$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left\{ \varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} \right\} - \frac{E\alpha\Delta\theta}{1-2\nu}\delta_{ij}$$

Linear Momentum: $\frac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j = \rho \frac{\partial^2 u_j}{\partial t^2}$

Boundary Conditions: $u_i = u_i^*(t)$ on S_1 $\qquad \sigma_{ij} n_i = t_j^*(t)$ on S_2

Simplifying Governing Equations

LMB in terms of $\underline{u}$: Cauchy-Navier equation

Eliminate σ, ε ; Note also $C_{ijkl}\varepsilon_{kl} = C_{ijkl}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$

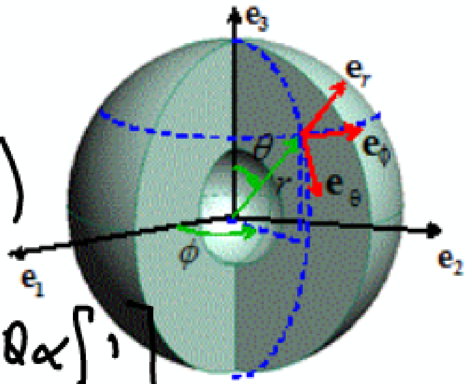
But $C_{ijkl} = C_{ijlk} \Rightarrow C_{ijkl}\varepsilon_{kl} = C_{ijkl}\frac{\partial u_k}{\partial x_l}$

Hence $$\boxed{\frac{\partial}{\partial x_j}\left\{C_{ijkl}\frac{\partial u_k}{\partial x_l}\right\} + \rho b_i = \rho\frac{\partial^2 u_i}{\partial t^2}}$$

For isotropic homogeneous material ; no temp change

$$\boxed{\begin{gathered}\frac{1}{1-2\nu}\frac{\partial^2 u_j}{\partial x_j \partial x_i} + \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\rho}{\mu} b_i = \frac{\rho}{\mu}\frac{\partial^2 u_i}{\partial t^2} \\ \frac{1}{1-2\nu}\nabla\nabla\cdot\underline{u} + \nabla^2\underline{u} + \frac{\rho}{\mu}\underline{b} = \frac{\rho}{\mu}\frac{\partial^2 \underline{u}}{\partial t^2}\end{gathered}}$$

Example: Spherical symmetry



Strain-displacement $\varepsilon = \frac{\partial u}{\partial r}\, \underline{e_r} \otimes \underline{e_r} + \frac{u_r}{r}\left(\underline{e_\theta} \otimes \underline{e_\theta} + \underline{e_\phi} \otimes \underline{e_\phi}\right)$

Stress-strain law

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \end{bmatrix} = \frac{2\mu}{1-2\nu} \begin{bmatrix} 1-\nu & 2\nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial u_r}{\partial r} \\ \frac{u_r}{r} \end{bmatrix} - \frac{2\mu(1+\nu)\Delta\Theta\alpha}{1-2\nu} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

LMB:
$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) + \rho b_r = \rho \frac{\partial^2 u_r}{\partial t^2}$$

Combine:
$$\frac{d}{dr}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right)\right) = \frac{\alpha(1+\nu)}{1-\nu}\frac{\partial \Delta\Theta}{\partial r} + \frac{1-2\nu}{2\mu(1-\nu)}\left[\frac{\partial^2 u_r}{\partial t^2} - \rho b_r\right]$$

Boundary conditions: prescribe u_r or σ_{rr} $\Big\}$ $r=a$, $r=b$

A solid, spherical nuclear fuel pellet with outer radius a is subjected to a uniform internal distribution of heat due to a nuclear reaction. The heating induces a steady-state temperature field

$$\theta(r) = \Delta\theta \frac{r^2}{a^2} + \theta_0$$

Calculate the distribution of stress in the pellet.

Procedure: (1) Solve ODE for $u(r)$ using $u(0)=0$
– 1 arbitrary constant

(2) Solve for constant using $\sigma_{rr} = 0$ @ $r = a$

Solution: $\sigma_{rr} = +4\alpha\Delta\theta\,\mu \dfrac{(a^2-r^2)\,(1+\nu)}{5a^2(1-\nu)}$

$\sigma_{\theta\theta} = \dfrac{4\alpha\Delta\theta\,\mu\,(a^2-2r^2)}{5a^2(1-\nu)}$

```
assume(-1<`&nu;`<1/2);
assume(EE>0);
assume(a>0);
assume(r>0);
T := DT*r^2/a^2+T0:
ode1 := diff(diff(r^2*uu(r),r)/r^2,r) = `&alpha;`*(1+`&nu;`)/(1-`&nu;`)*diff(T,r);
```

$$\frac{2\,\mathrm{uu}(r)+r^2\,\frac{\partial^2}{\partial r^2}\,\mathrm{uu}(r)+4\,r\,\frac{\partial}{\partial r}\,\mathrm{uu}(r)}{r^2}-\frac{2\left(2\,r\,\mathrm{uu}(r)+r^2\,\frac{\partial}{\partial r}\,\mathrm{uu}(r)\right)}{r^3}=-\frac{2\,\alpha\,\mathrm{DT}\,r\,(\nu+1)}{a^2\,(\nu-1)}$$

```
u := simplify(solve(ode({ode1,uu(0)=0}, uu(r))))[1]
```

$$-\frac{r\,(5\,\nu\,C8\,a^2-5\,C8\,a^2+3\,\alpha\,\mathrm{DT}\,r^2+3\,\alpha\,\nu\,\mathrm{DT}\,r^2)}{15\,a^2\,(\nu-1)}$$

```
srr := eval(EE/(1+`&nu;`)/(1-2*`&nu;`)
*((1-`&nu;`)*diff(u,r) + 2*`&nu;`*u/r) - EE*`&alpha;`*T/(1-2*`&nu;`)):
C8val := solve(subs(srr,r=a),C8):
srrfull := simplify(subs(srr,C8=C8val))
```

$$\left\{-\frac{2\,\alpha\,\mathrm{DT}\,\mathrm{EE}\,(a^2-r^2)}{5\,a^2\,(\nu-1)}\right\}$$

```
sqq := eval(EE/(1+`&nu;`)/(1-2*`&nu;`)
*(`&nu;`*diff(u,r) + u/r) - EE*`&alpha;`*T/(1-2*`&nu;`)):
sqqfull := simplify(subs(sqq,C8=C8val))
```

$$\left\{-\frac{2\,\alpha\,\mathrm{DT}\,\mathrm{EE}\,(a^2-2\,r^2)}{5\,a^2\,(\nu-1)}\right\}$$

Potential Solutions for 3D static linear elasticity

Goal: Simplify Navier equation

Assumptions: Constant temperature
Isotropic
Homogeneous

Define Papkovich-Neuber potentials $\underline{\psi}$, ϕ satisfying

$$\nabla^2 \underline{\psi} = -\rho \underline{b} \qquad \nabla^2 \phi = -\underline{x}\cdot\underline{b}$$

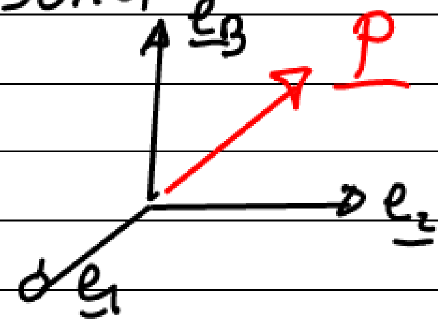
Then define: $\underline{u} = \dfrac{1}{\mu}\left(\underline{\psi} + \dfrac{1}{4(1-\nu)} \nabla(\phi - \underline{x}\cdot\underline{\psi})\right)$

Then C-N equation satisfied automatically
(This representation is complete)

Also useful :

$$2(1-\nu)\sigma_{ij} = 2\nu \frac{\partial \psi_k}{\partial x_k}\delta_{ij} + (1-2\nu)\left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right) + x_k \frac{\partial^2 \psi_k}{\partial x_i \partial x_j} + \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

Example : Point force in an infinite solid (acting at origin)



Point force is a Dirac Delta distribution of body force

$$\rho \underline{b}(\underline{x}) = \underline{P}\,\delta(\underline{x})$$

$\delta(\underline{x})$ is a delta distribution with property

$$\int_V \delta(\underline{x})\, f(\underline{x})\, dV = f(\underline{0})$$

Hence we seek $\underline{\psi}$, ϕ satisfying

$$\nabla^2 \underline{\psi} = -\delta(\underline{x})\,\underline{P} \qquad \nabla^2 \phi = 0$$

Guess $\underline{\psi} = G(r)\,\underline{P}$ $\qquad \phi = 0$

Substitute and integrate over sphere radius r surrounding origin

$$\underline{P}\int_V \nabla \cdot \nabla G(r)\, dV = -\underline{P}\int_V \delta(\underline{x})\, dV = -\underline{P}$$

Div theorem $\quad \underline{P}\int_A \nabla G(r) \cdot \underline{n}\, dA = -\underline{P}$

$$4\pi r^2 \frac{dG}{dr} = -1 \quad \Rightarrow \quad G = \frac{1}{4\pi r} \qquad r = \sqrt{x_k x_k}$$

Hence $\psi = \dfrac{P}{4\pi r}$ $\qquad \phi = 0$

Hence $u_i = \dfrac{1}{16\pi(1-\nu).r}\left\{ \dfrac{P_k x_k x_i}{r^2} + (3-4\nu) P_i \right\}$

$$\sigma_{ij} = \frac{-1}{8\pi(1-\nu) r}\left\{ \frac{3 P_k x_k x_i x_j}{r^3} + \frac{(1-2\nu)}{r}\left(P_i x_j + P_j x_i - P_k x_k \delta_{ij} \right) \right\}$$