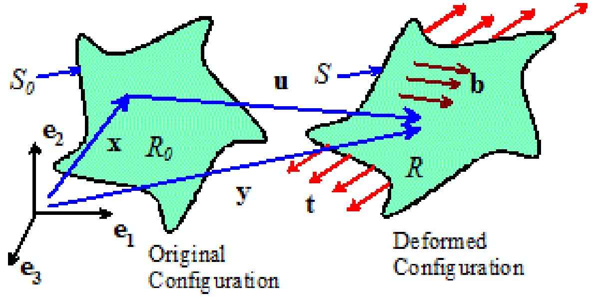
Review: Cauchy-Navier equation

Governing equation for displacements in linear elastic solid:

$$\frac{\partial}{\partial x_j}\left\{C_{ijkl}\left(\frac{\partial u_k}{\partial x_l}-\alpha_{kl}\Delta\theta\right)\right\}+\rho b_i=\rho\frac{\partial^2 u_i}{\partial t^2}$$

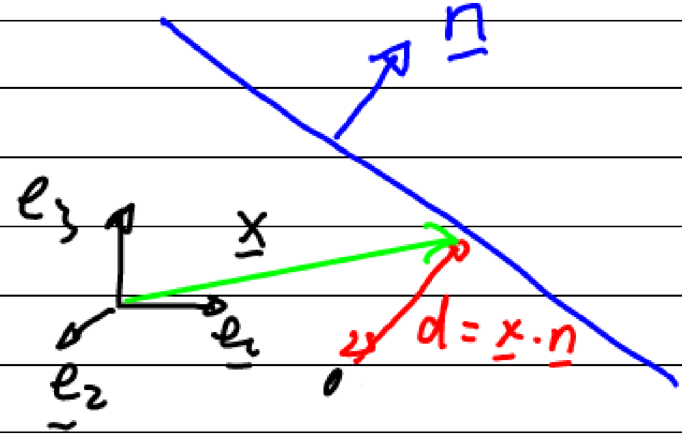


Special Case: Isotropy, homogeneous, constant temperature

$$\frac{1}{1-2\nu}\frac{\partial^2 u_j}{\partial u_j \partial x_i}+\frac{\partial^2 u_i}{\partial u_j \partial x_j}+\frac{\rho}{\mu}b_i=\frac{\rho}{\mu}\frac{\partial^2 u_i}{\partial t^2}$$

Solutions to dynamic problems in linear elasticity

Plane wave solutions



Guess solution of form

$$u_i = a_i f\left(t - \frac{\underline{x}\cdot\underline{n}}{c}\right)$$

C-N equation $\quad C_{ijkl}\dfrac{\partial u_k}{\partial x_j \partial x_l} = \rho \dfrac{\partial^2 u_i}{\partial t^2}$

$$\left(\frac{1}{c^2} C_{ijkl}\, n_l n_j - \rho\delta_{ik}\right) a_k\, f''\left(t - \frac{\underline{x}\cdot n}{c}\right) = 0$$

Define $\quad A_{ik} = C_{ijkl}\, n_l n_j$

$$A_{ik} a_k = \rho c^2 a_i$$

Hence ρc^2 are eigenvalues of acoustic tensor

a_i " eigenvectors

Hence for any $\underline{n}$; 3 wave speeds
Each wave speed has a characteristic a_i

Special case : Isotropic material

$$C_{ijkl} = \mu (\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + \frac{2\mu\nu}{1-2\nu} \delta_{ij}\delta_{kl}$$

$$A_{ik} = \mu n_i n_k + \mu \delta_{ik} + \frac{2\mu\nu}{1-2\nu} n_i n_k$$

$$A_{ik} = \frac{\mu}{1-2\nu} n_i n_j + \mu \delta_{ik}$$

Hence $\left(\frac{\mu}{1-2\nu} n_i n_k + \delta_{ik} \mu \right) a_k = \rho c^2 a_i$

Solutions:

① $\underline{a} = \underline{n}$ $= \sqrt{\frac{1-\nu}{1-2\nu} \frac{\mu}{\rho}} = c_L$ – "P-wave" Longitudinal wave

② $\underline{a} \cdot \underline{n} = 0$ $\sqrt{\frac{\mu}{\rho}} = c_S$ – "S-wave" shear wave

Stresses : $\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{2\mu\nu}{1-2\nu}\delta_{ij}\frac{\partial u_k}{\partial x_k}$

For p wave

$$\sigma_{ij} = -\left(\mu\left(n_i n_j + n_j n_i\right) + \frac{2\mu\nu}{1-2\nu}\delta_{ij}\right)\frac{1}{c_L^2} f''\left(t - \frac{\underline{x}\cdot\underline{n}}{c_L}\right)$$

For S-wave

$$\sigma_{ij} = -\left(\mu\left(a_i n_j + a_j n_i\right)\right)\frac{1}{c_S^2} f''\left(t - \frac{\underline{x}\cdot\underline{n}}{c_S}\right)$$

(shear stress in basis parallel & perpendicular to $\underline{n}$

Potential representations for dynamic solutions.

Let $\underline{\Psi}$, ϕ be a vector and scalar potential

satisfying $\nabla \cdot \underline{\Psi} = 0$ $\quad c_s^2 \nabla^2 \underline{\Psi} = \frac{\partial^2 \underline{\Psi}}{\partial t^2}$

$$c_L^2 \nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2}$$

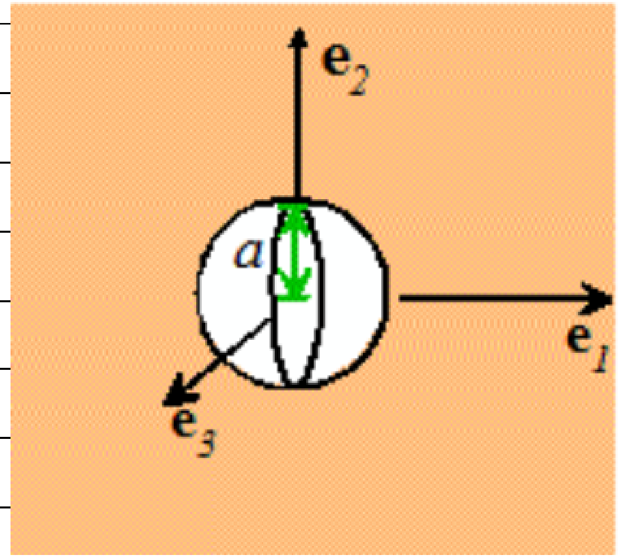
Then set $\underline{u} = \nabla \phi + \nabla \times \underline{\Psi}$ $\quad *$

$$\frac{1}{2\mu}\sigma_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{1}{2}\left(\epsilon_{ilk}\frac{\partial^2 \Psi_k}{\partial x_l \partial x_j} + \epsilon_{jlk}\frac{\partial^2 \Psi_k}{\partial x_l \partial x_i} \right) + \frac{\nu \delta_{ij}}{1-2\nu}\frac{\partial^2 \phi}{\partial x_k \partial x_k}$$

Proof: substitute $*$ into isotropic C-N equation, simplify example: for P-wave : $\phi = f\left(t - \frac{\underline{x}\cdot\underline{n}}{c_L}\right)$

For S-wave $\psi_i = b_i f\left(t - \frac{\underline{x}\cdot\underline{n}}{c_s}\right)$

Example: dynamic solution to a pressurized cavity



Solid is at rest for $t<0$

For time $t>0$ const pressure p acts on $r=a$

Expect p-wave with spherical front

$\underline{\psi} = 0 \qquad \phi = \frac{c}{r} f\left(t - \frac{(r-a)}{c_L}\right)$

Can show any ϕ of this form satisfies governing eq for ϕ

To find f, use $\sigma_{ij}\dfrac{x_j}{a} = p\dfrac{x_i}{a}$

This yields a 2^{nd} order ODE $f(t)$

Solution for $f(t)$ is a damped harmonic oscillation.

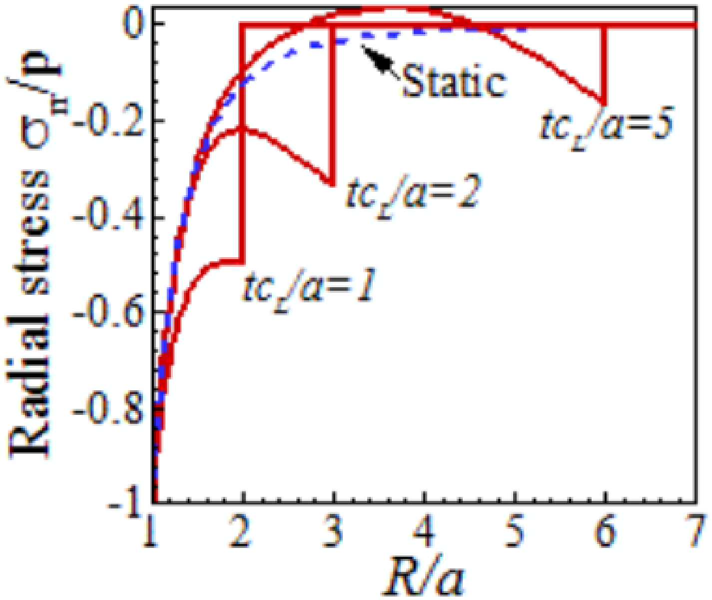
$$\Psi_i = 0 \quad \phi = -\frac{(1+\nu)a^3 p}{2Er}\left\{1-\sqrt{2(1-\nu)}e^{-\alpha s}\sin\left(\beta s+\gamma\right)\right\}$$

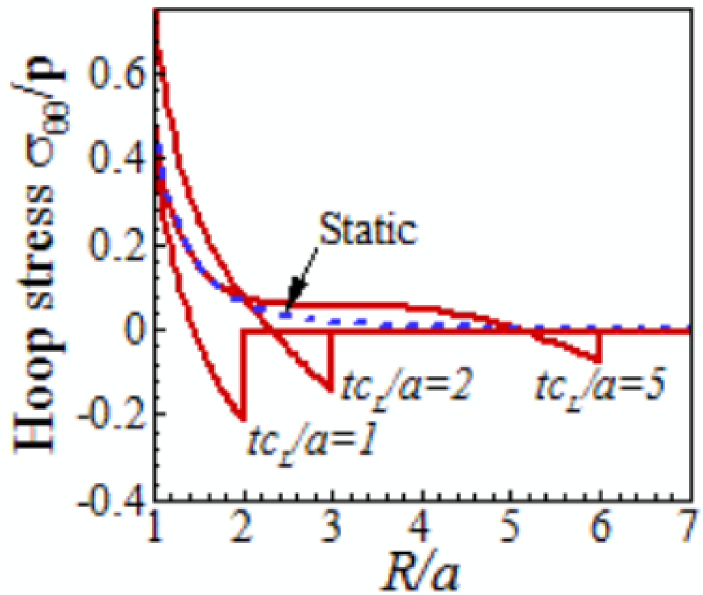
$$\alpha = \frac{1-2\nu}{1-\nu} \qquad \beta = \frac{\sqrt{1-2\nu}}{1-\nu} \qquad \gamma = \cot^{-1}\sqrt{1-2\nu} \qquad r = \sqrt{x_k x_k}$$

$$s = \begin{cases} (c_L t - r + a)/a & r-a < c_L t \\ 0 & r-a > c_L t \end{cases}$$

$$u_i = \frac{(1+\nu)a^3 p x_i}{2ER^3}\left\{1-\sqrt{2(1-\nu)}e^{-\alpha s}\sin(\beta s+\gamma)\left(\frac{\beta R\cot(\beta s+\gamma)-\alpha R}{a}+1\right)\right\}$$

$$\sigma_{ij} = -\frac{a^3 p}{2R^3}\left(3\frac{x_i x_j}{R^2}-\delta_{ij}\right)\left\{1-\sqrt{2(1-\nu)}e^{-\alpha s}\sin(\beta s+\gamma)\left(\frac{\beta R\cot(\beta s+\gamma)-\alpha R}{a}+1\right)\right\}$$
$$+\frac{ap}{2R}\left(\frac{x_i x_j}{R^2}+\frac{\nu\delta_{ij}}{1-2\nu}\right)\sqrt{2(1-\nu)}e^{-\alpha s}\sin(\beta s+\gamma)\left\{(\alpha^2-\beta^2)-2\beta\alpha\cot(\beta s+\gamma)\right\}$$





Notes: for short times solution is approx planar p-wave
Solution behind front relaxes to static state
Dynamic solution differs significantly from static sol